Dynamische und regionale Ozeanographie WS 2015/16

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11 – Waves and Instabilities

Recapitulation
- Layered models
- Quasi-geostrophic approximation
- Potential vorticity
- Geostrophic adjustment

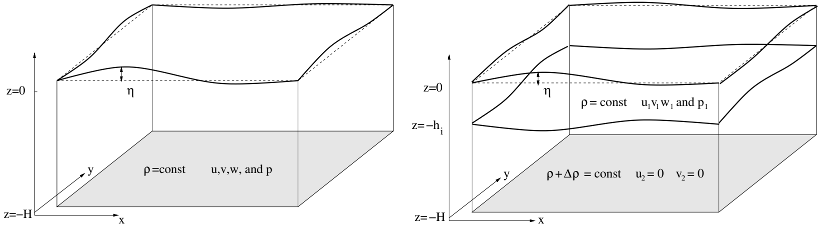
Waves
- Rossby waves

- "barotropic" and "baroclinic" layered model

$$\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \to g\Delta\rho/\rho_0$ ("baroclinic")



- consider the barotropic or baroclinic layered model

$$\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{Dh}{Dt} + (\not{\eta} + H)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

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- take curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and with $\beta = \partial f/\partial y$

- consider the barotropic or baroclinic layered model

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- assume small Rossby number Ro, i.e. dominant geostrophic balance

$v \approx (g/f)\,\partial h/\partial x$, $u \approx -(g/f)\,\partial h/\partial y$, $\zeta \approx (g/f)\,(\partial^2 h/\partial x^2 + \partial^2 h/\partial y^2)$

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- using this yields the quasi-geostrophic potential vorticity equation

$$\frac{D}{Dt}\left(\nabla^2 h - R^{-2} h\right) + \beta\frac{\partial h}{\partial x} \approx 0$$

with the "Rossby radius" $R = \sqrt{gH}/|f|$

- quasi-geostrophic potential vorticity (PV) equation

$$\frac{D}{Dt}\left(\nabla^2 h - R^{-2}h\right) + \beta\frac{\partial h}{\partial x} = 0 \quad , \quad \frac{D}{Dt}\left(\zeta - (f_0/H)h + f_0 + \beta y\right) = 0$$

valid for $Ro \ll 1$ and $L \ll a$

with the "Rossby radius" $R = \sqrt{gH}/|f|$ and Earth radius a

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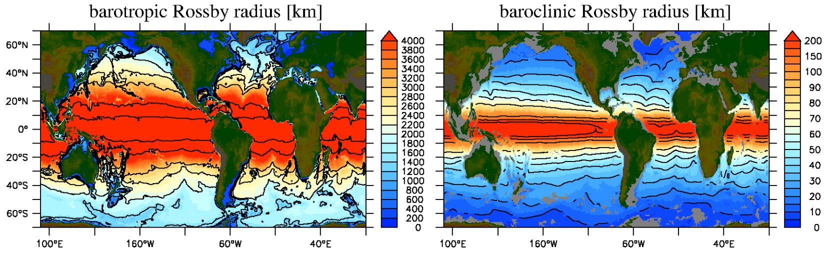
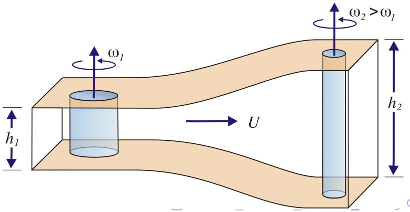


- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

q is conserved for fluid parcels in single layer



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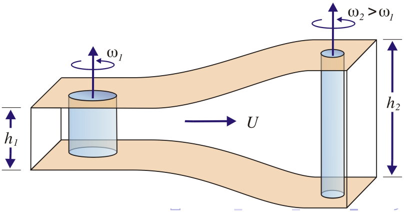
- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \zeta - \frac{f_0}{H} h + f$$

q is (approximately) conserved in single layer for $Ro \ll 1$



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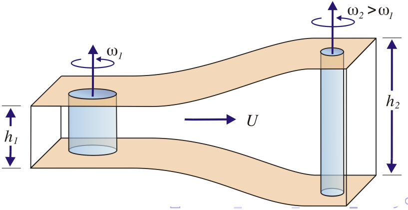
- $\zeta = (g/f)\nabla^2 h$ is relative vorticity
- $-(f_0/H)h$ is stretching vorticity
- $f = f_0 + \beta y$ is planetary vorticity



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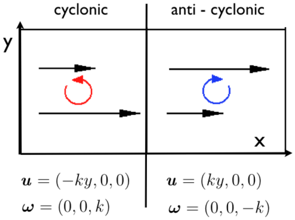
  q is conserved for fluid parcels in single layer

- $h = const$, ζ initially zero, parcel moves northward

  f increases but $q = (f + \zeta)/h$ has to stay constant

  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ decreases $\rightarrow$ anticyclonic rotation



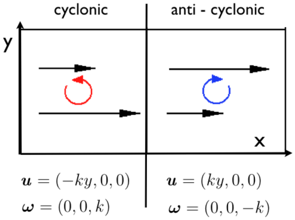
$u = -ay \;,\;\; v = 0 \;\; \rightarrow \;\; \zeta = a > 0$: cyclonic (anticlockwise) rotation

$u = +ay \;,\;\; v = 0 \;\; \rightarrow \;\; \zeta = a < 0$: anticyclonic (clockwise) rotation

- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q = \zeta - \frac{f_0}{H} h + f$$

  q is conserved for fluid parcels in single layer
- $h = const$, ζ initially zero, parcel moves northward
  f increases but $q = (f + \zeta)/h$ has to stay constant
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ decreases $\rightarrow$ anticyclonic rotation
- $h = const$, ζ initially zero, parcel moves southward
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ increases $\rightarrow$ more cyclonic rotation



$u = -ay$, $v = 0$ $\rightarrow$ $\zeta = a > 0$: cyclonic (anticlockwise) rotation
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- potential vorticity equation for a single layer

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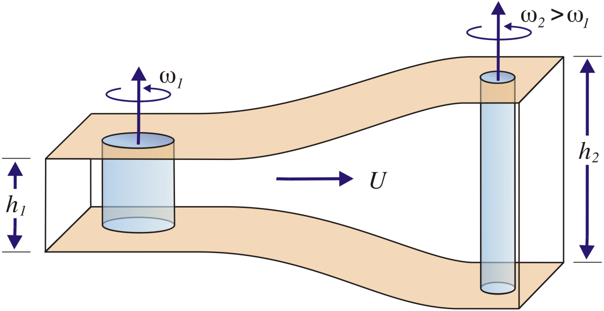
  q is conserved for fluid parcels in single layer

- $f = const$, ζ initially zero, parcel moves to deeper water
  $\rightarrow \zeta = \partial v / \partial x - \partial u / \partial y$ increases $\rightarrow$ cyclonic rotation



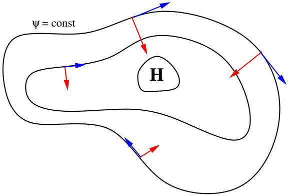
- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \zeta - \frac{f_0}{H} h + f$$

q is (approximately) conserved in single layer for $Ro \ll 1$

- $\psi = gh/f_0$ is streamfunction for the quasi-geostrophic flow

$$u \approx -\frac{g}{f_0}\frac{\partial h}{\partial y} = -\frac{\partial \psi}{\partial y} \quad , \quad v \approx \frac{g}{f_0}\frac{\partial h}{\partial x} = \frac{\partial \psi}{\partial x}$$



$$\boldsymbol{u} = \begin{pmatrix} -\partial\psi/\partial y \\ \partial\psi/\partial x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ 0 \end{pmatrix} = \boldsymbol{k} \times \boldsymbol{\nabla}\psi$$

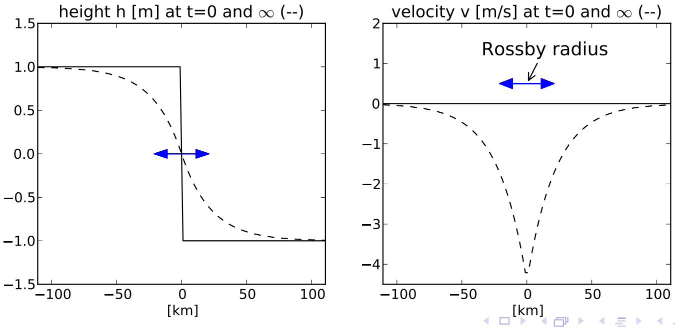
- $\boldsymbol{u}$ (blue arrow): anti-clockwise rotation of $\boldsymbol{\nabla}\psi$ (red arrow) by 90^o

- potential vorticity $q = \nabla^2 h - h/R^2$ (for $f = const$) stays constant
- initial and steady state solution of h are given by

$$h|_{t=0} = \begin{cases} h_0, & \text{if } x < 0 \\ -h_0, & \text{if } x > 0 \end{cases}, \quad h|_{\infty} = \begin{cases} h_0(1 - e^{x/R}), & \text{if } x < 0 \\ -h_0(1 - e^{-x/R}), & \text{if } x > 0 \end{cases}$$

with Rossby radius $R = \sqrt{gH}/|f|$

- Gravity waves establish geostrophic balance on scale larger than R



- potential vorticity equation for layered model

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \zeta - \frac{f_0}{H} h + f_0 + \beta y$$

with relative vorticity $\zeta = (g/f_0)\boldsymbol{\nabla}^2 h$

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- linearized version $(D/Dt \to \partial/\partial t)$

$$\frac{D}{Dt}\left(\frac{g}{f_0}\boldsymbol{\nabla}^2 h - \frac{f_0}{H}h + f_0 + \beta y\right) \quad =$$

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$$\frac{\partial}{\partial t}\left(\frac{g}{f_0}\boldsymbol{\nabla}^2 h - \frac{f_0}{H} h\right) + \beta \frac{g}{f_0}\frac{\partial h}{\partial x} \approx 0$$

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with Rossby radius $R = \sqrt{gH}/|f|$

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with Rossby radius $R = \sqrt{gH}/|f|$

- "vorticity wave" $h = A \exp i(k_1 x + k_2 y - \omega t) = A \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$

$$(-i\omega)\left((i\boldsymbol{k})^2 A \exp i(..) - R^{-2} A \exp i(..)\right) + \beta(ik_1) A \exp i(..) = 0$$

- potential vorticity equation for layered model

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$$(-i\omega)\left((i\boldsymbol{k})^2 A\exp i(..) - R^{-2}A\exp i(..)\right) + \beta(ik_1)A\exp i(..) = 0$$

$$-\omega\left(-(\boldsymbol{k})^2 - R^{-2}\right) + \beta k_1 = 0$$

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}}$$

with $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- "vorticity wave" → Rossby wave dispersion relation

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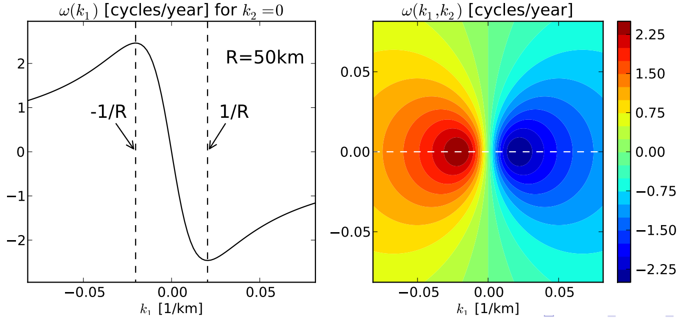
with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$



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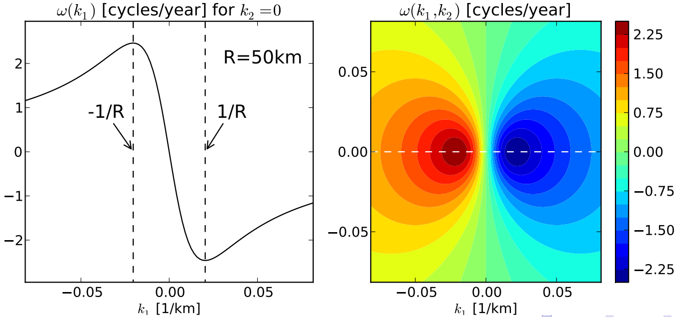
- slow, only present with planetary vorticity gradient $df/dy = \beta$



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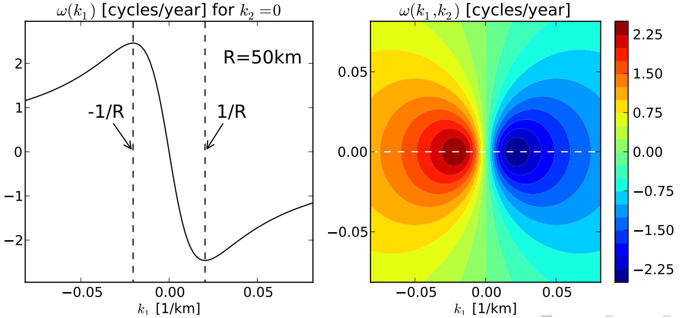
  with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$
- slow, only present with planetary vorticity gradient $df/dy = \beta$
- for $k_1 > 0$ phase propagation speed $c = \omega/k$ is negative
  for $k_1 < 0$ phase propagation speed $c = \omega/k$ is positive
  $\rightarrow$ phase propagation is always westward



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- meridional group velocity of Rossby waves

$$c_g^y = \frac{\partial \omega}{\partial k_2} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_2$$

- "vorticity wave" $\rightarrow$ Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- meridional group velocity of Rossby waves

$$c_g^y = \frac{\partial \omega}{\partial k_2} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_2 = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

- "vorticity wave" $\rightarrow$ Rossby wave dispersion relation

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- zonal group velocity of Rossby waves

$$c_g^x = \frac{\partial \omega}{\partial k_1} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_1 - \frac{\beta}{k^2 + R^{-2}}$$

- "vorticity wave" $\to$ Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- meridional group velocity of Rossby waves

$$c_g^y = \frac{\partial \omega}{\partial k_2} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_2 = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

- zonal group velocity of Rossby waves

$$\begin{aligned} c_g^x &= \frac{\partial \omega}{\partial k_1} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_1 - \frac{\beta}{k^2 + R^{-2}} \\ &= \frac{2\beta k_1^2}{(k^2 + R^{-2})^2} - \frac{\beta (k^2 + R^{-2})}{(k^2 + R^{-2})^2} \end{aligned}$$

- "vorticity wave" $\rightarrow$ Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

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$$c_g^y = \frac{\partial \omega}{\partial k_2} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_2 = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

- zonal group velocity of Rossby waves

$$\begin{aligned} c_g^x &= \frac{\partial \omega}{\partial k_1} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_1 - \frac{\beta}{k^2 + R^{-2}} \\ &= \frac{2\beta k_1^2}{(k^2 + R^{-2})^2} - \frac{\beta(k^2 + R^{-2})}{(k^2 + R^{-2})^2} = \frac{2\beta k_1^2 - \beta k_1^2 - \beta k_2^2 - \beta R^{-2}}{(k^2 + R^{-2})^2} \end{aligned}$$

- "vorticity wave" $\rightarrow$ Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- meridional group velocity of Rossby waves

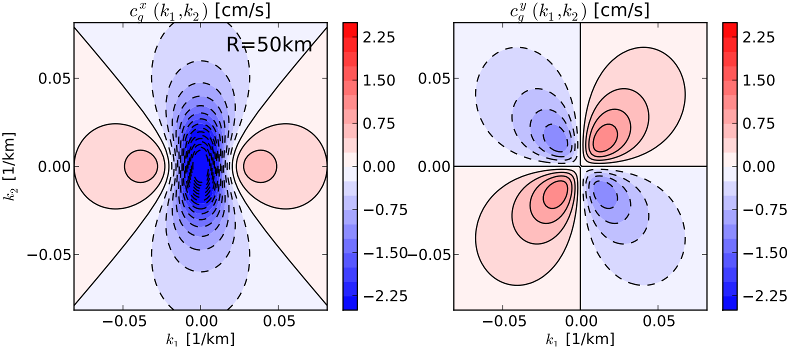
$$c_g^y = \frac{\partial \omega}{\partial k_2} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_2 = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

- zonal group velocity of Rossby waves

$$\begin{aligned} c_g^x &= \frac{\partial \omega}{\partial k_1} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_1 - \frac{\beta}{k^2 + R^{-2}} \\ &= \frac{2\beta k_1^2}{(k^2 + R^{-2})^2} - \frac{\beta(k^2 + R^{-2})}{(k^2 + R^{-2})^2} = \frac{2\beta k_1^2 - \beta k_1^2 - \beta k_2^2 - \beta R^{-2}}{(k^2 + R^{-2})^2} \\ &= \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \end{aligned}$$

- Group velocity of Rossby waves

$$c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$



- direction of c_g^x westward for small k_1, eastward for large k_1
- for $k_2 = 0$ $c_g^x = 0$ at $k_1 = \pm 1/R \rightarrow \omega_{max} = -\beta R/2$
- direction of c_g^y always opposite to k_2

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{\left(k^2 + R^{-2}\right)^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{\left(k^2 + R^{-2}\right)^2}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- long wave limit for $\lambda \gg R$ or $k \ll 1/R$ or $kR \to 0$

$$\omega \stackrel{kR\to 0}{=} -\beta k_1 R^2 \quad , \quad c_g^x \stackrel{kR\to 0}{=} -\beta R^2 \quad , \quad c_g^y \stackrel{kR\to 0}{=} 0$$

$\to$ westward phase and energy propagation

$\to$ no dispersion: $c \stackrel{kR\to 0}{=} c_g^x$ (for a wave with $k_2 = 0$)

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

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$\to$ no dispersion: $c \overset{kR\to 0}{=} c_g^x$ (for a wave with $k_2 = 0$)

- start again with PV equation and neglect relative vorticity $\zeta = \boldsymbol{\nabla}^2 h$

$$\frac{\partial}{\partial t}\left(\cancel{\boldsymbol{\nabla}^2 h} - R^{-2} h\right) + \beta \frac{\partial h}{\partial x} = 0$$

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

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$$\frac{\partial}{\partial t}\left(\cancel{\boldsymbol{\nabla}^2 h} - R^{-2} h\right) + \beta \frac{\partial h}{\partial x} = 0$$

- wave ansatz $h = A \exp i(k_1 x - \omega t)$ yields

$$(-i\omega)(-R^{-2} A \exp i(\ldots)) + \beta(ik_1) A \exp i(\ldots) = 0 \quad \rightarrow \quad \omega = -\beta k_1 R^2$$

$\rightarrow$ long wave limit is identical to vanishing relative vorticity

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

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- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- short wave limit for $\lambda \ll R$ or $k \gg 1/R$ or $kR \to \infty$

$$\omega \overset{kR\to\infty}{=} -\frac{\beta k_1}{k^2} \quad , \quad c_g^x \overset{kR\to\infty}{=} \beta \frac{k_1^2 - k_2^2}{k^4} \quad , \quad c_g^y \overset{kR\to\infty}{=} \frac{2\beta k_1 k_2}{k^4}$$

$\rightarrow$ eastward energy propagation (for $k_1^2 > k_2^2$), dispersion

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- short wave limit for $\lambda \ll R$ or $k \gg 1/R$ or $kR \to \infty$

$$\omega \overset{kR\to\infty}{=} -\frac{\beta k_1}{k^2} \quad , \quad c_g^x \overset{kR\to\infty}{=} \beta \frac{k_1^2 - k_2^2}{k^4} \quad , \quad c_g^y \overset{kR\to\infty}{=} \frac{2\beta k_1 k_2}{k^4}$$

$\rightarrow$ eastward energy propagation (for $k_1^2 > k_2^2$), dispersion

- start again with PV equation and neglect stretching vorticity

$$\frac{\partial}{\partial t} \left(\boldsymbol{\nabla}^2 h - \cancel{R^{-2} h} \right) + \beta \frac{\partial h}{\partial x} = 0$$

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- short wave limit for $\lambda \ll R$ or $k \gg 1/R$ or $kR \to \infty$

$$\omega \overset{kR\to\infty}{=} -\frac{\beta k_1}{k^2} \quad , \quad c_g^x \overset{kR\to\infty}{=} \beta \frac{k_1^2 - k_2^2}{k^4} \quad , \quad c_g^y \overset{kR\to\infty}{=} \frac{2\beta k_1 k_2}{k^4}$$

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- wave ansatz $h = A \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$ yields

$$(-i\omega)(i\boldsymbol{k})^2 A \exp i(\ldots) + \beta(ik_1) A \exp i(\ldots) = 0 \quad \rightarrow \quad \omega = -\beta k_1/k^2$$

$\rightarrow$ short wave limit is identical to vanishing stretching vorticity

- long wave limit for $\lambda \gg R$ or $k \ll 1/R$ or $kR \to 0$

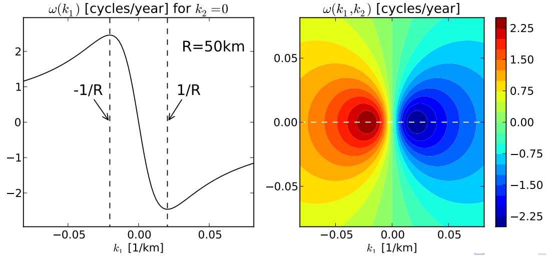
$$\omega \overset{kR\to 0}{=} -\beta k_1 R^2 \quad , \quad c_g^x \overset{kR\to 0}{=} -\beta R^2 \quad , \quad c_g^y \overset{kR\to 0}{=} 0$$

$\to$ westward phase and energy propagation, no dispersion, vanishing relative vorticity

- short wave limit for $\lambda \ll R$ or $k \gg 1/R$ or $kR \to \infty$

$$\omega \overset{kR\to \infty}{=} -\frac{\beta k_1}{k^2} \quad , \quad c_g^x \overset{kR\to \infty}{=} \beta \frac{k_1^2 - k_2^2}{k^4} \quad , \quad c_g^y \overset{kR\to \infty}{=} \frac{2\beta k_1 k_2}{k^4}$$

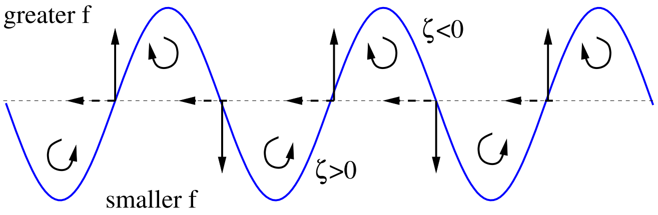
$\to$ westward phase but eastward energy propagation, dispersion, vanishing stretching vorticity



- (short) Rossby wave propagation mechanism

$$q \approx f + \zeta = const \quad , \quad \text{vanishing stretching vorticity}$$

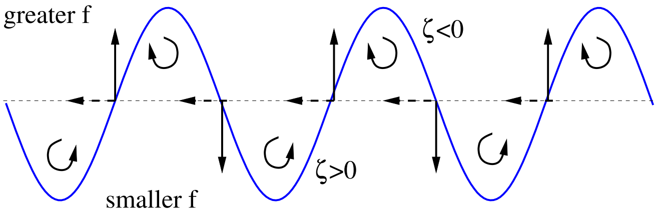
- for northward parcel displacement f increase $\rightarrow \zeta < 0$
- for southward parcel displacement f decrease $\rightarrow \zeta > 0$



- (short) Rossby wave propagation mechanism

$$q \approx f + \zeta = const \quad , \quad \text{vanishing stretching vorticity}$$

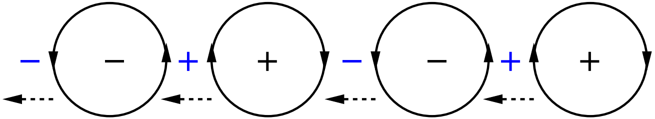
- for northward parcel displacement f increase $\rightarrow \zeta < 0$
- for southward parcel displacement f decrease $\rightarrow \zeta > 0$
- parcels in between without initial displacement
  will be shifted to north or south by relative vorticity
  $\rightarrow$ westward phase propagation in both cases



- (long) Rossby wave propagation mechanism

$$q \approx f - f_0 h/H = const \quad , \quad \text{vanishing relative vorticity}$$

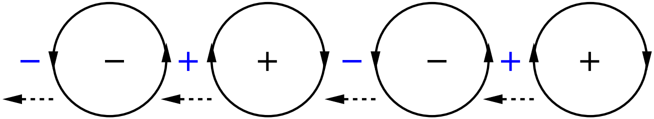
- for northward parcel displacement f increase $\rightarrow h > 0$ (for $f > 0$)
- for southward parcel displacement f decrease $\rightarrow h < 0$



- (long) Rossby wave propagation mechanism

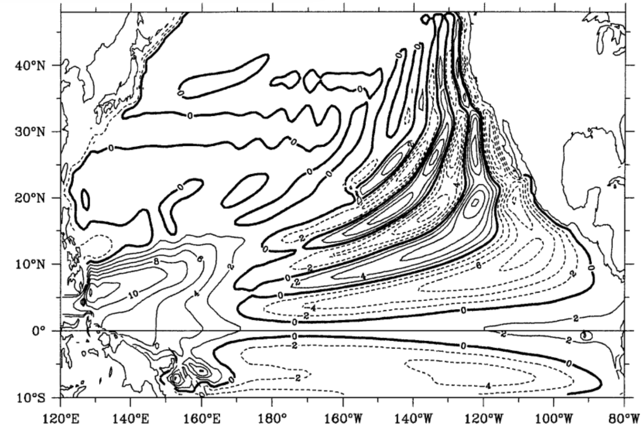
$$q \approx f - f_0 h/H = const \quad , \quad \text{vanishing relative vorticity}$$

- for northward parcel displacement f increase $\rightarrow h > 0$ (for $f > 0$)
- for southward parcel displacement f decrease $\rightarrow h < 0$
- northward geostrophic flow west of $h > 0$ and east of $h < 0$
  generates increase in $f \rightarrow$ increase in h (blue)
  southward geostrophic flow east of $h > 0$ and west of $h < 0$
  generates decrease in $f \rightarrow$ decrease in h (blue)
  $\rightarrow$ westward phase propagation in both cases



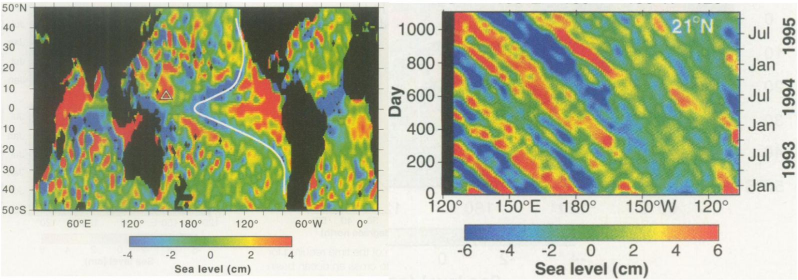
Movie

- Rossby waves in a simple Pacific Ocean model setting up the large-scale circulation



from Qui et al (1997)

- Rossby waves in satellite altimeter data
  snapshot (left) and along 21°N in the Pacific Ocean (right)



from Chelton and Schlax (1996)