11 – Waves and Instabilities

Waves
  Rossby waves

- potential vorticity equation for layered model

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \zeta - \frac{f_0}{H}h + f_0 + \beta y$$

with relative vorticity $\zeta = (g/f_0)\nabla^2 h$

- linearized version $(D/Dt \to \partial/\partial t)$

$$\frac{D}{Dt}\left(\frac{g}{f_0}\nabla^2 h - \frac{f_0}{H}h + f_0 + \beta y\right) \quad = \quad \frac{D}{Dt}\left(\frac{g}{f_0}\nabla^2 h - \frac{f_0}{H}h\right) + \beta v = 0$$

$$\frac{\partial}{\partial t}\left(\frac{g}{f_0}\nabla^2 h - \frac{f_0}{H}h\right) + \beta \frac{g}{f_0}\frac{\partial h}{\partial x} \approx 0 \quad \to \quad \frac{\partial}{\partial t}\left(\nabla^2 h - R^{-2}h\right) + \beta\frac{\partial h}{\partial x} = 0$$

with Rossby radius $R = \sqrt{gH}/|f|$

- "vorticity wave" $h = A \exp i(k_1 x + k_2 y - \omega t) = A \exp i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)$

$$(-i\omega)\left((i\boldsymbol{k})^2 A \exp i(..) - R^{-2} A \exp i(..)\right) + \beta(ik_1) A \exp i(..) = 0$$

$$-\,\omega\left(-(\boldsymbol{k})^2 - R^{-2}\right) + \beta k_1 = 0$$

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}}$$

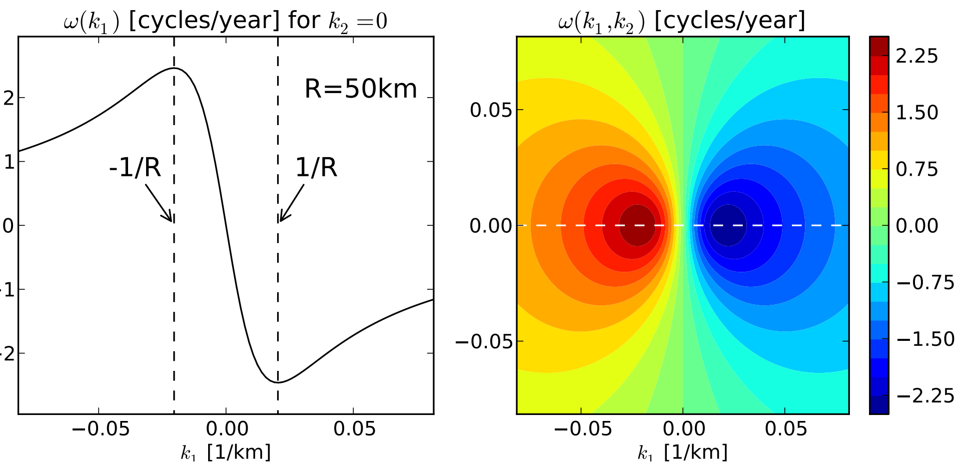
with $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- "vorticity wave" $\rightarrow$ Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- slow, only present with planetary vorticity gradient $df/dy = \beta$
- for $k_1 > 0$ phase propagation speed $c = \omega/k$ is negative
  for $k_1 < 0$ phase propagation speed $c = \omega/k$ is positive
  $\rightarrow$ phase propagation is always westward



- "vorticity wave" $\rightarrow$ Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- meridional group velocity of Rossby waves

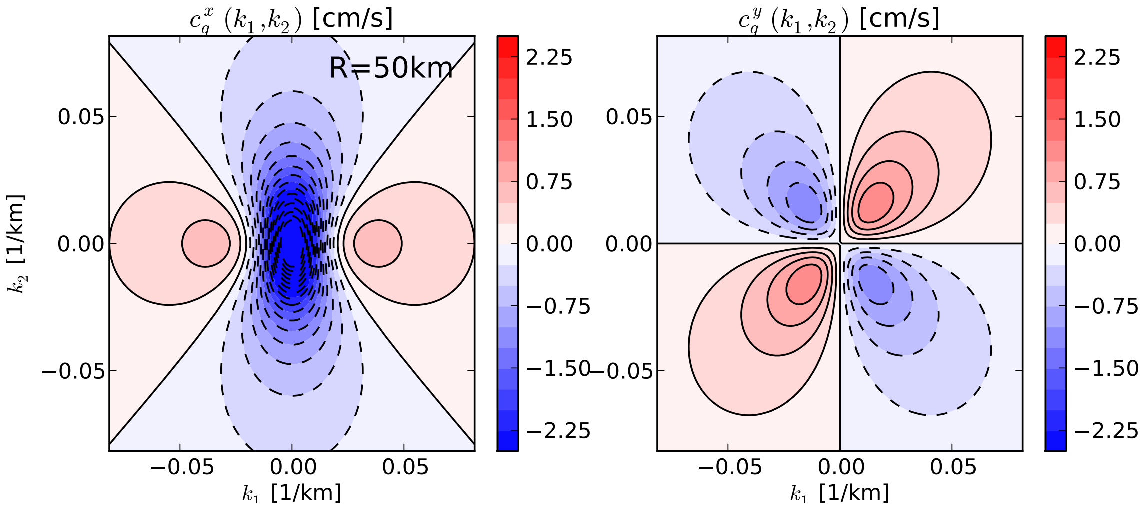
$$c_g^y = \frac{\partial \omega}{\partial k_2} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_2 = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

- zonal group velocity of Rossby waves

$$\begin{aligned} c_g^x &= \frac{\partial \omega}{\partial k_1} = -(-1)\frac{\beta k_1}{(k^2 + R^{-2})^2} 2k_1 - \frac{\beta}{k^2 + R^{-2}} \\ &= \frac{2\beta k_1^2}{(k^2 + R^{-2})^2} - \frac{\beta(k^2 + R^{-2})}{(k^2 + R^{-2})^2} = \frac{2\beta k_1^2 - \beta k_1^2 - \beta k_2^2 - \beta R^{-2}}{(k^2 + R^{-2})^2} \\ &= \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \end{aligned}$$

- Group velocity of Rossby waves

$$c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$



- direction of c_g^x westward for small k_1, eastward for large k_1
- for $k_2 = 0$ $c_g^x = 0$ at $k_1 = \pm 1/R \rightarrow \omega_{max} = -\beta R/2$
- direction of c_g^y always opposite to k_2

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- long wave limit for $\lambda \gg R$ or $k \ll 1/R$ or $kR \rightarrow 0$

$$\omega \overset{kR\rightarrow 0}{=} -\beta k_1 R^2 \quad , \quad c_g^x \overset{kR\rightarrow 0}{=} -\beta R^2 \quad , \quad c_g^y \overset{kR\rightarrow 0}{=} 0$$

$\rightarrow$ westward phase and energy propagation

$\rightarrow$ no dispersion: $c \overset{kR\rightarrow 0}{=} c_g^x$ (for a wave with $k_2 = 0$)

- start again with PV equation and neglect relative vorticity $\zeta = \boldsymbol{\nabla}^2 h$

$$\frac{\partial}{\partial t}\left(\cancel{\boldsymbol{\nabla}^2 h} - R^{-2} h\right) + \beta \frac{\partial h}{\partial x} = 0$$

- wave ansatz $h = A \exp i(k_1 x - \omega t)$ yields

$(-i\omega)(-R^{-2} A \exp i(\ldots)) + \beta(ik_1) A \exp i(\ldots) = 0 \quad \rightarrow \quad \omega = -\beta k_1 R^2$

$\rightarrow$ long wave limit is identical to vanishing relative vorticity

- Rossby wave dispersion relation

$$\omega = -\frac{\beta k_1}{k^2 + R^{-2}} \quad , \quad c_g^x = \beta \frac{k_1^2 - k_2^2 - R^{-2}}{(k^2 + R^{-2})^2} \quad , \quad c_g^y = \frac{2\beta k_1 k_2}{(k^2 + R^{-2})^2}$$

with Rossby radius $R = \sqrt{gH}/|f|$ and $k^2 = k_1^2 + k_2^2 = (\boldsymbol{k})^2$

- short wave limit for $\lambda \ll R$ or $k \gg 1/R$ or $kR \to \infty$

$$\omega \overset{kR\to\infty}{=} -\frac{\beta k_1}{k^2} \quad , \quad c_g^x \overset{kR\to\infty}{=} \beta \frac{k_1^2 - k_2^2}{k^4} \quad , \quad c_g^y \overset{kR\to\infty}{=} \frac{2\beta k_1 k_2}{k^4}$$

$\to$ eastward energy propagation (for $k_1^2 > k_2^2$), dispersion

- start again with PV equation and neglect stretching vorticity

$$\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}^2 h - \cancel{R^{-2}h}\right) + \beta \frac{\partial h}{\partial x} = 0$$

- wave ansatz $h = A \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$ yields

$(-i\omega)(i\boldsymbol{k})^2 A \exp i(\ldots) + \beta(ik_1)A \exp i(\ldots) = 0 \quad \to \quad \omega = -\beta k_1/k^2$

$\to$ short wave limit is identical to vanishing stretching vorticity

- long wave limit for $\lambda \gg R$ or $k \ll 1/R$ or $kR \to 0$

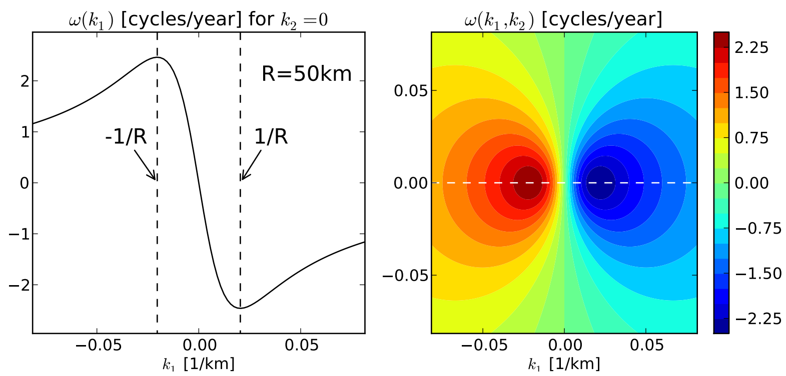
$$\omega \overset{kR\to 0}{=} -\beta k_1 R^2 \quad , \quad c_g^x \overset{kR\to 0}{=} -\beta R^2 \quad , \quad c_g^y \overset{kR\to 0}{=} 0$$

$\to$ westward phase and energy propagation, no dispersion, vanishing relative vorticity

- short wave limit for $\lambda \ll R$ or $k \gg 1/R$ or $kR \to \infty$

$$\omega \overset{kR\to\infty}{=} -\frac{\beta k_1}{k^2} \quad , \quad c_g^x \overset{kR\to\infty}{=} \beta \frac{k_1^2 - k_2^2}{k^4} \quad , \quad c_g^y \overset{kR\to\infty}{=} \frac{2\beta k_1 k_2}{k^4}$$

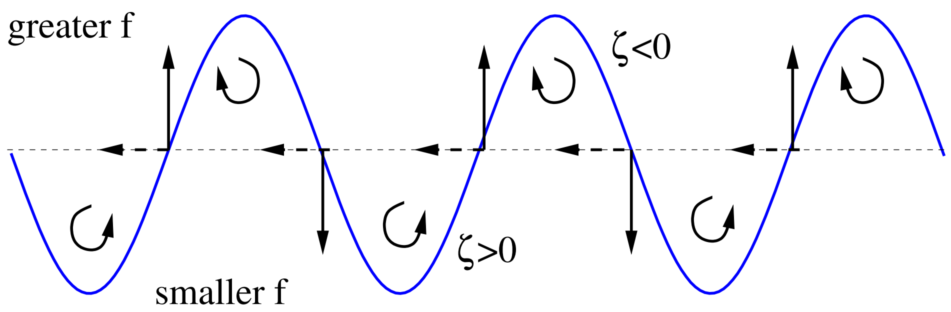
$\to$ westward phase but eastward energy propagation, dispersion, vanishing stretching vorticity



- (short) Rossby wave propagation mechanism

$$q \approx f + \zeta = const \quad , \quad \text{vanishing stretching vorticity}$$

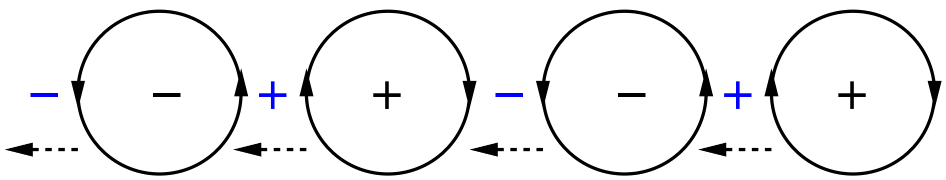
- for northward parcel displacement f increase $\rightarrow \zeta < 0$
- for southward parcel displacement f decrease $\rightarrow \zeta > 0$
- parcels in between without initial displacement
  will be shifted to north or south by relative vorticity
  $\rightarrow$ westward phase propagation in both cases



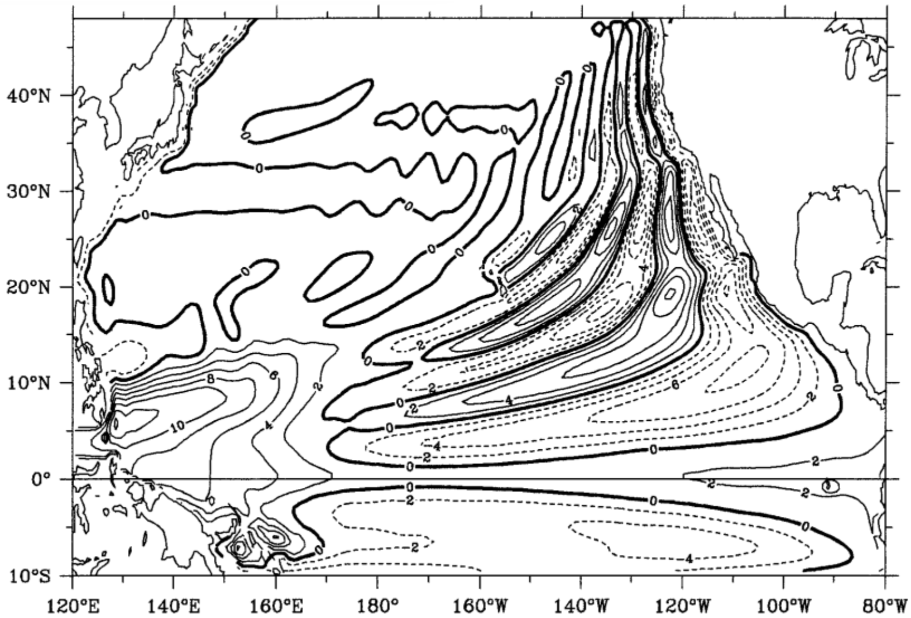
- (long) Rossby wave propagation mechanism

$$q \approx f - f_0 h/H = const \quad , \quad \text{vanishing relative vorticity}$$

- for northward parcel displacement f increase $\rightarrow h > 0$ (for $f > 0$)
- for southward parcel displacement f decrease $\rightarrow h < 0$
- northward geostrophic flow west of $h > 0$ and east of $h < 0$
  generates increase in f $\rightarrow$ increase in h (blue)
  southward geostrophic flow east of $h > 0$ and west of $h < 0$
  generates decrease in f $\rightarrow$ decrease in h (blue)
  $\rightarrow$ westward phase propagation in both cases

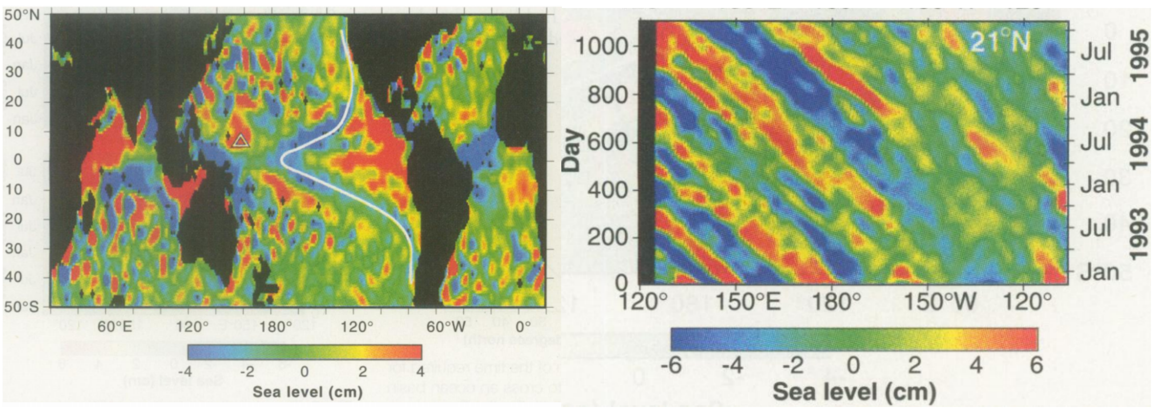


- Rossby waves in a simple Pacific Ocean model setting up the large-scale circulation



from Qui et al (1997)

- Rossby waves in satellite altimeter data
  snapshot (left) and along 21°N in the Pacific Ocean (right)



from Chelton and Schlax (1996)