Dynamische und regionale Ozeanographie WS 2015/16

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11 – Waves and Instabilities

Recapitulation

- Layered models
- Gravity waves without rotation
- Gravity waves with rotation

Waves

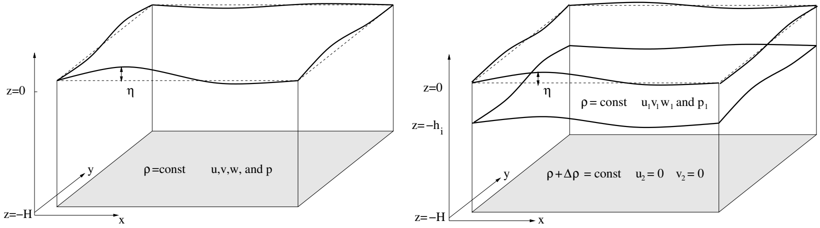
- Kelvin waves
- Quasi-geostrophic approximation
- Potential vorticity
- Geostrophic adjustment

- "barotropic" and "baroclinic" layered model

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \rightarrow g\Delta\rho/\rho_0$ ("baroclinic")



- consider the (linearized) layered model with $f = 0$
  but now include y dependency $\rightarrow$ plane wave

$$\frac{\partial u}{\partial t} - \cancel{f}v = -g\frac{\partial h}{\partial x}\ ,\ \frac{\partial v}{\partial t} + \cancel{f}u = -g\frac{\partial h}{\partial y}\ ,\ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

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- combine momentum and thickness equation to wave equation

$$\frac{\partial \boldsymbol{u}}{\partial t} = - \quad g\boldsymbol{\nabla} h\ , \qquad \frac{\partial h}{\partial t} + \quad H\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

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$$\boldsymbol{\nabla}\cdot\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla}\cdot g\boldsymbol{\nabla}h\ ,\ \frac{\partial}{\partial t}\frac{\partial h}{\partial t} + \frac{\partial}{\partial t}H\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\ \rightarrow\ \frac{\partial^2 h}{\partial t^2} - gH\boldsymbol{\nabla}^2 h = 0$$

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- wave solution $h = A\exp i(k_1 x + k_2 y - \omega t) = A\exp i(\boldsymbol{k} \cdot \boldsymbol{x}_h - \omega t)$

$$\frac{\partial h}{\partial t} = -i\omega A\exp i(\ldots)\quad ,\ \frac{\partial^2 h}{\partial t^2} = (i\omega)^2 A\exp i(\ldots) = -\omega^2 A\exp i(\ldots)$$

  with wavenumber vector $\boldsymbol{k} = (k_1, k_2)$

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$$\nabla h = i\boldsymbol{k} A \exp i(\ldots)\quad ,\ \nabla \cdot \nabla h = i^2 \boldsymbol{k} \cdot \boldsymbol{k} A \exp i(\ldots) = -k^2 A \exp i(\ldots)$$

with wavenumber vector $\boldsymbol{k} = (k_1, k_2)$ and $k = |\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2}$

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$$\boldsymbol{\nabla}h = i\boldsymbol{k}A\exp i(\ldots) \quad , \ \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}h = i^2\boldsymbol{k}\cdot\boldsymbol{k}A\exp i(\ldots) = -k^2 A\exp i(\ldots)$$

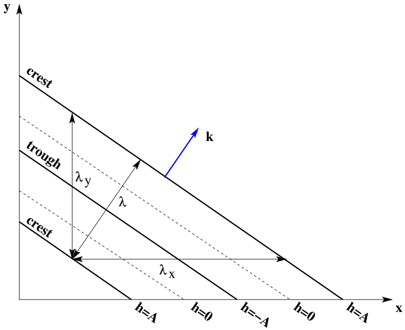
  with wavenumber vector $\boldsymbol{k} = (k_1, k_2)$ and $k = |\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2}$

- this works as long as

$$-\omega^2\exp i(..) + k^2 gH\exp i(..) = 0 \ \rightarrow \ \omega^2 = k^2 gH \ \rightarrow \ \omega = \pm k\sqrt{gH}$$

  which is still the dispersion relation for a gravity wave (for $f = 0$)

- plane wave in two dimensions is given by $h = A \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$
- wavenumber vector $\boldsymbol{k}$ gives direction of phase propagation



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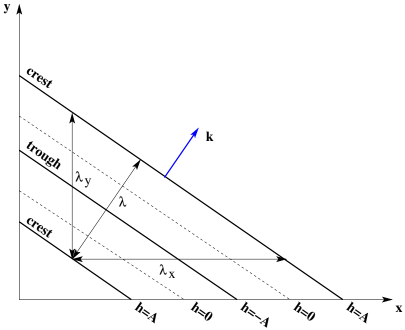
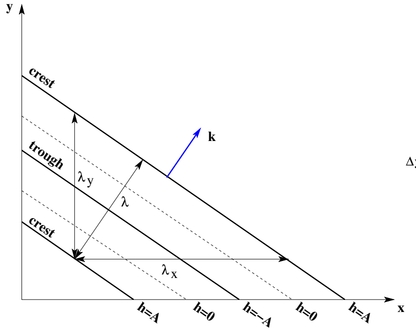
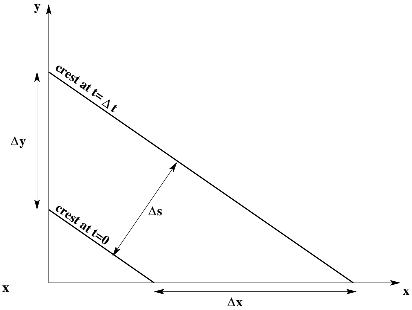
- wavelength $\lambda = 2\pi/k = 2\pi/\sqrt{k_1^2 + k_2^2}$



- plane wave in two dimensions is given by $h = A \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$
- wavenumber vector $\mathbf{k}$ gives direction of phase propagation
- wavelength $\lambda = 2\pi/k = 2\pi/\sqrt{k_1^2 + k_2^2}$
- phase propagates from $t = 0$ to $t = \Delta t$ the distance $\Delta s = c\Delta t$
  $\rightarrow$ phase velocity c in two dimensions





- add two waves with different $\boldsymbol{k}$ and ω but same amplitude

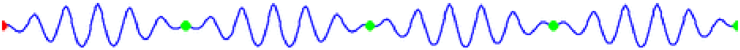
$$h = A\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) + A\cos(\boldsymbol{k}'\cdot\boldsymbol{x} - \omega' t)$$

$$h \approx 2A\cos\left(\frac{\Delta\boldsymbol{k}}{2}\cdot[\boldsymbol{x} - \boldsymbol{c}_g t]\right)\cos\left(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t\right)$$

  with the wavenumber difference $\Delta\boldsymbol{k} = \boldsymbol{k}' - \boldsymbol{k}$

  and the *group velocity* $\boldsymbol{c}_g = \left(\frac{\partial\omega}{\partial k_1}, \frac{\partial\omega}{\partial k_2}\right) = \partial\omega/\partial\boldsymbol{k}$

- amplitude modulation with speed $\boldsymbol{c}_g$ and wave length $\Delta\boldsymbol{k}$



- $\boldsymbol{c}_g$ is the speed at which the amplitudes (energy) propagates
- while c is the propagation speed of the phase (in the direction $\boldsymbol{k}$)
- both are in general different and different from particle velocity

- thickness, curl and divergence for $f = const$

$$\begin{aligned}
\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \\
\frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \\
\frac{\partial \xi}{\partial t} - f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= -g\boldsymbol{\nabla}^2 h
\end{aligned}$$

with $\zeta = \partial v/\partial x - \partial u/\partial y$ and $\xi = \partial u/\partial x + \partial v/\partial y$

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- time differentiate divergence and replace with curl and thickness eq.

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\frac{\partial^2 \xi}{\partial t^2} + f^2\left(\xi - (gH/f^2)\boldsymbol{\nabla}^2\xi\right) &= 0
\end{aligned}$$

- thickness, curl and divergence for $f = const$

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with Rossby radius $R = \sqrt{gH}/|f|$

- combined thickness, curl and divergence eq. for $f = const$

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- look for wave solutions

$$\xi(x, y, t) = \xi_0 \exp i(k_1 x + k_2 y - \omega t)$$

with complex constant ξ_0

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with complex constant ξ_0 which yields

$$(-i\omega)^2 \xi_0 \exp(\ldots) + f^2 \left(1 - R^2 (ik_1)^2 - R^2 (ik_2)^2\right) \xi_0 \exp(\ldots) \quad = \quad 0$$

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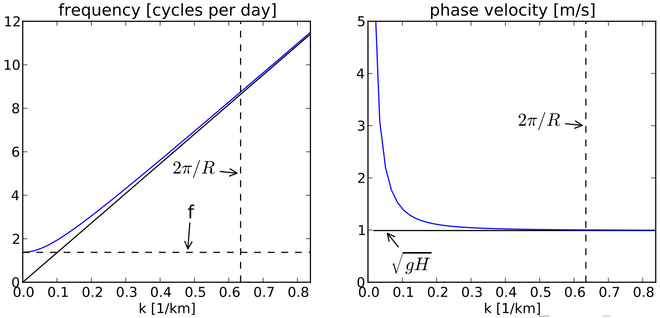
- this is a (plane wave) solution as long as ω satisfies

$$\omega = \pm\sqrt{f^2 \left(1 + R^2 k^2\right)}$$

with $k^2 = |\boldsymbol{k}|^2 = k_1^2 + k_2^2$

- gravity wave dispersion relation ($f \neq 0$ in blue, $f = 0$ in black)

$$\omega = \pm\sqrt{f^2\left(1 + R^2k^2\right)} \quad , \quad c = \pm\sqrt{f^2\left(1/k^2 + R^2\right)}$$



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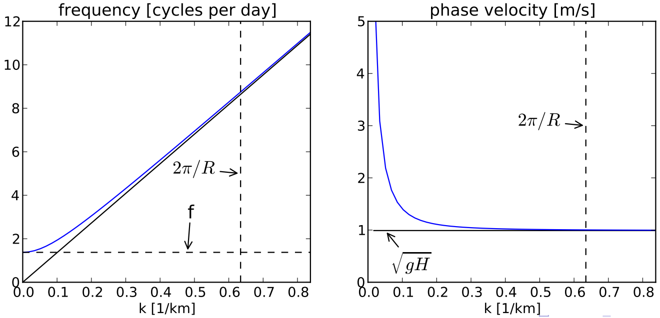
- different phase velocity $c = \omega/k$ for different $\boldsymbol{k}$ $\rightarrow$ dispersive wave



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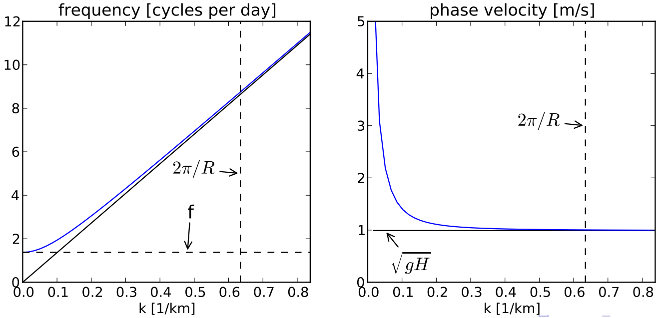
- different phase velocity $c = \omega/k$ for different $\boldsymbol{k} \rightarrow$ dispersive wave
- short wave limit for $\lambda = 2\pi/k \ll R \rightarrow R^2k^2 \gg 1$

$$\omega \overset{Rk\to\infty}{=} \pm\sqrt{f^2R^2k^2} = \pm k\sqrt{gH}$$



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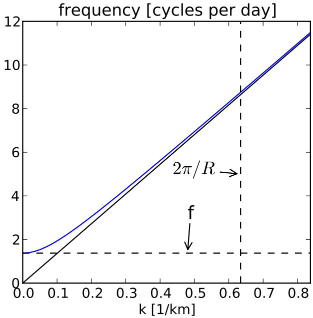
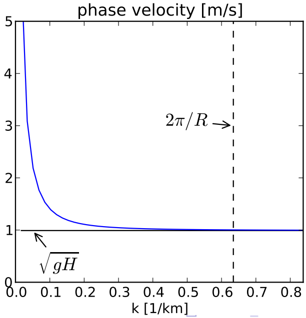
- different phase velocity $c = \omega/k$ for different $\boldsymbol{k}$ $\rightarrow$ dispersive wave
- short wave limit for $\lambda = 2\pi/k \ll R$ $\rightarrow$ $R^2k^2 \gg 1$

$$\omega \stackrel{Rk\to\infty}{=} \pm\sqrt{f^2R^2k^2} = \pm k\sqrt{gH} \quad , \quad c \stackrel{Rk\to\infty}{=} \pm\sqrt{gH}$$

$\rightarrow$ (non-dispersive) gravity waves without rotation (black lines)





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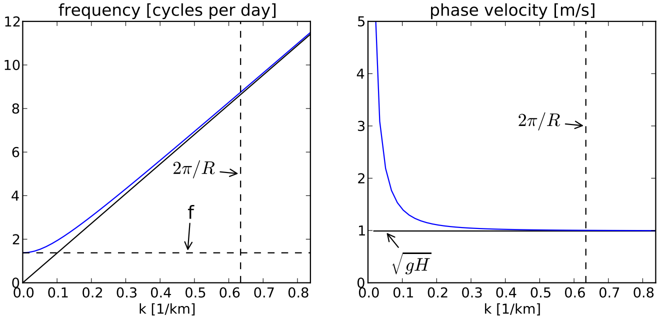
- different phase velocity $c = \omega/k$ for different $\boldsymbol{k}$ $\rightarrow$ dispersive wave



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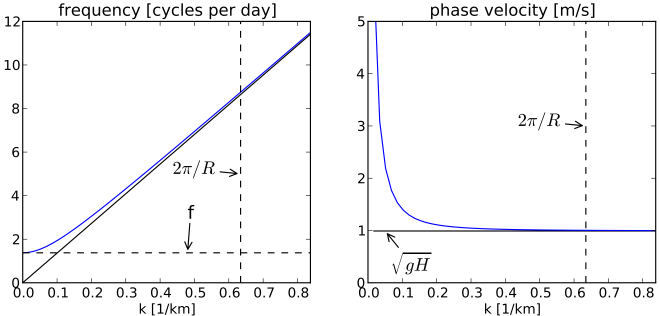
- different phase velocity $c = \omega/k$ for different $\boldsymbol{k}$ $\rightarrow$ dispersive wave
- long wave limit for $\lambda = 2\pi/k \gg R \rightarrow R^2k^2 \ll 1$

$$\omega \overset{Rk\rightarrow 0}{=} \pm f \quad , \quad c \overset{Rk\rightarrow 0}{=} \pm\infty$$



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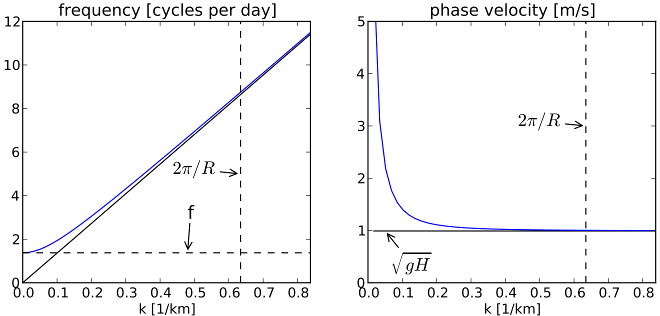
- different phase velocity $c = \omega/k$ for different $\boldsymbol{k}$ $\rightarrow$ dispersive wave
- long wave limit for $\lambda = 2\pi/k \gg R \rightarrow R^2k^2 \ll 1$

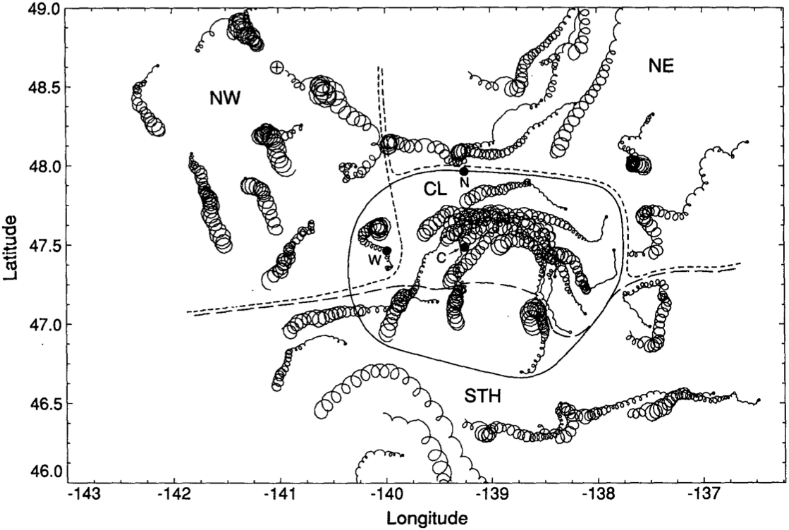
$$\omega \overset{Rk\rightarrow 0}{=} \pm f \quad , \quad c \overset{Rk\rightarrow 0}{=} \pm\infty$$

- these are inertial oscillations which also result from

$$\partial u/\partial t - fv = 0 \quad , \quad \partial v/\partial t + fu = 0$$



- trajectories of surface drifter → inertial oscillations

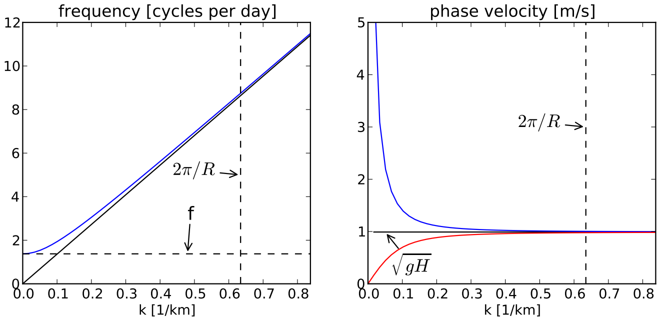


from d'Asaro et al 1995

- gravity wave dispersion relation ($f \neq 0$ in blue, $f = 0$ in black)

$$\omega = \pm\sqrt{f^2\left(1 + R^2 k^2\right)}$$

- group velocity is given by $\boldsymbol{c}_g = (gH/\omega)\boldsymbol{k}$ (red line for $f \neq 0$)



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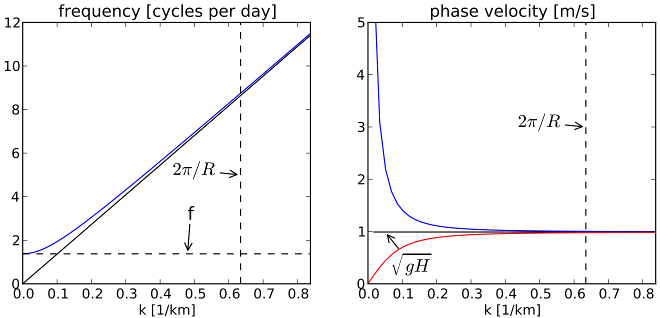

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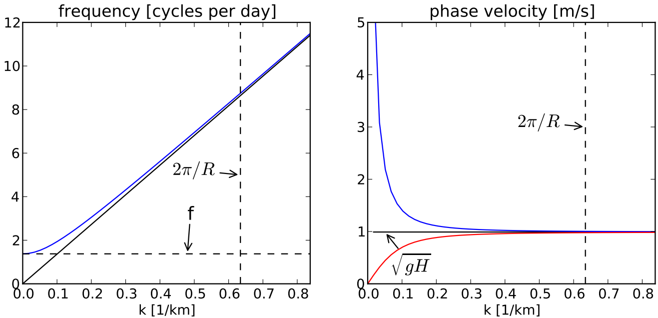

- consider again the (linearized) layered model with $f \neq 0$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}\ ,\ \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}\ ,\ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

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- for y dependency of A we consider the second equation

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$$fu = -g\frac{\partial h}{\partial y} \;\rightarrow\; (f/c)A = -A' \;\rightarrow\; A = A_0 e^{-f\,y/c} = A_0 e^{\pm y/R}$$

with $c = \omega/k = \pm\sqrt{gH}$ and with Rossby radius $R = \sqrt{gH}/|f|$

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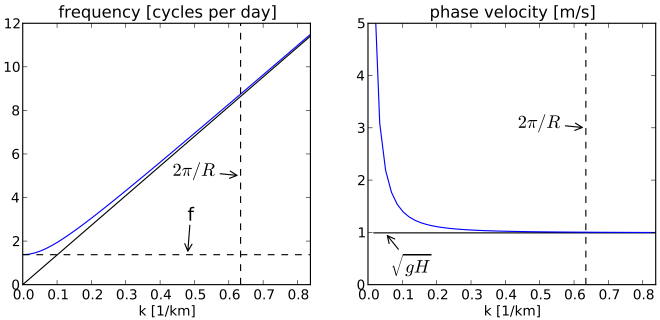
- only the decaying solution in y is reasonable
- Kelvin wave

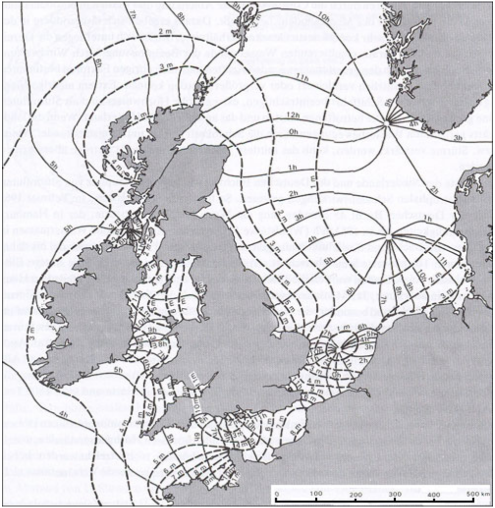
- Kelvin wave along solid boundary at $y = 0$

  $$h = A_0 e^{\pm y/R} \exp i(kx - \omega t) \ , \ u = (gA_0/c) e^{\pm y/R} \exp i(kx - \omega t) \ , \ v = 0$$

  and $\omega = \pm k\sqrt{gH}$ and with Rossby radius $R = \sqrt{gH}/|f|$
- only the decaying solution in y is reasonable
- works in the same way for boundary along x or any other direction



- tidal Kelvin wave in the North Sea



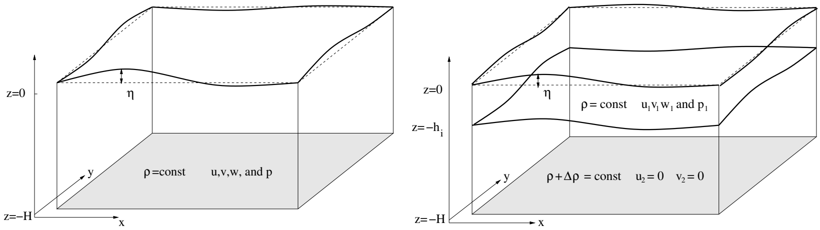
from Klett (2014)

- "barotropic model" and "baroclinic model"

$$\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \to g\Delta\rho/\rho_0$ ("baroclinic")



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with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and with $\beta = \partial f/\partial y$

- assume small Rossby number Ro, i.e. dominant geostrophic balance

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assuming $|(g\beta/f^2)\,\partial h/\partial y| \ll |\partial v/\partial x|$, i.e small variations of f

$$\frac{g}{f^2}\frac{\partial h}{\partial y}\frac{\partial f}{\partial y} \sim \frac{U}{\Omega}\frac{\Omega}{a}$$

with Earth radius a

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with Earth radius a , i.e. only length scales L smaller than a are valid

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with the "Rossby radius" $R = \sqrt{gH}/|f|$ and Earth radius a

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- single equation in h: quasi-geostrophic potential vorticity equation valid for $Ro \ll 1$ and $L \ll a$

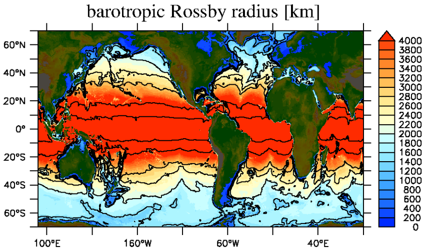
- quasi-geostrophic potential vorticity (PV) equation

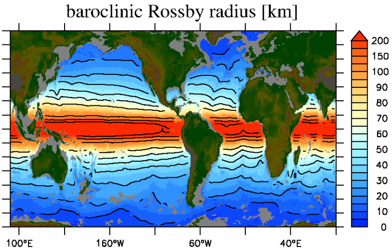
$$\frac{\partial}{\partial t}\left(\nabla^2 h - R^{-2} h\right) + \beta \frac{\partial h}{\partial x} = 0$$

valid for $Ro \ll 1$ and $L \ll a$

with the "Rossby radius" $R = \sqrt{gH}/|f|$ and Earth radius a

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \to g\Delta\rho/\rho_0$ ("baroclinic")





- consider again the layered model, but this time with $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

- take curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

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$$\begin{aligned}
\frac{\partial}{\partial x}(2.\text{eqn}) \; : \quad \frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}(\boldsymbol{u} \cdot \boldsymbol{\nabla} v) + \frac{\partial}{\partial x}(fu) &= -g \frac{\partial}{\partial x}\frac{\partial h}{\partial y} \\
\frac{\partial}{\partial y}(1.\text{eqn}) \; : \quad \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}(\boldsymbol{u} \cdot \boldsymbol{\nabla} u) - \frac{\partial}{\partial y}(fv) &= -g \frac{\partial}{\partial y}\frac{\partial h}{\partial x}
\end{aligned}$$

- consider again the layered model, but this time with $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

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\frac{\partial}{\partial y}(1.\text{eqn}) \; : \quad \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}(\boldsymbol{u} \cdot \boldsymbol{\nabla} u) - \frac{\partial}{\partial y}(fv) &= -g\frac{\partial}{\partial y}\frac{\partial h}{\partial x}
\end{aligned}$$

subtract both

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + \frac{\partial}{\partial x}(fu) + \frac{\partial}{\partial y}(fv) = 0$$

with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$

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- take curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\begin{aligned} \frac{\partial}{\partial x}(2.\mathrm{eqn}) \; : \quad \frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}(\boldsymbol{u} \cdot \boldsymbol{\nabla} v) + \frac{\partial}{\partial x}(fu) &= -g \frac{\partial}{\partial x}\frac{\partial h}{\partial y} \\ \frac{\partial}{\partial y}(1.\mathrm{eqn}) \; : \quad \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}(\boldsymbol{u} \cdot \boldsymbol{\nabla} u) - \frac{\partial}{\partial y}(fv) &= -g \frac{\partial}{\partial y}\frac{\partial h}{\partial x} \end{aligned}$$

subtract both

$$\begin{aligned} \frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + \frac{\partial}{\partial x}(fu) + \frac{\partial}{\partial y}(fv) &= 0 \\ \frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v &= 0 \end{aligned}$$

with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and with $\beta = \partial f/\partial y$

- curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v \quad = \quad 0$$

- calculate

$$\frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u \quad =$$

- curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \nabla v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \nabla u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

- calculate

$$\frac{\partial \boldsymbol{u}}{\partial x} \cdot \nabla v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \nabla u = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}$$

- curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + f \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \beta v = 0$$

- calculate

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \end{aligned}$$

- curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

- calculate

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \\
&= \frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
&= \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)
\end{aligned}$$

- curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

- calculate

$$\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ &= \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\end{aligned}$$

- now use $|\zeta| \ll |f| \rightarrow U/L \ll \Omega \rightarrow U/(\Omega L) = Ro \ll 1$

$$\frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v \approx 0$$

for $Ro \ll 1$

- assume again $Ro \ll 1$, i.e. dominant geostrophic balance

$$O(Ro) - fv = -g\frac{\partial h}{\partial x} \quad , \quad O(Ro) + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{Dh}{Dt} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- curl $\zeta \approx (g/f)\,(\partial^2 h/\partial x^2 + \partial^2 h/\partial y^2)$ for $Ro \ll 1$ and $L \ll a$

$$\frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v \approx 0$$

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$$\frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v \approx 0$$
$$\frac{D}{Dt}\left(\frac{g}{f}\boldsymbol{\nabla}^2 h\right) - \frac{f}{H}\frac{Dh}{Dt} + \beta\frac{g}{f}\frac{\partial h}{\partial x} \approx 0$$

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with the "Rossby radius" $R = \sqrt{gH}/|f|$ and Earth radius a

- assume again $Ro \ll 1$, i.e. dominant geostrophic balance

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$$\frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v \approx 0$$

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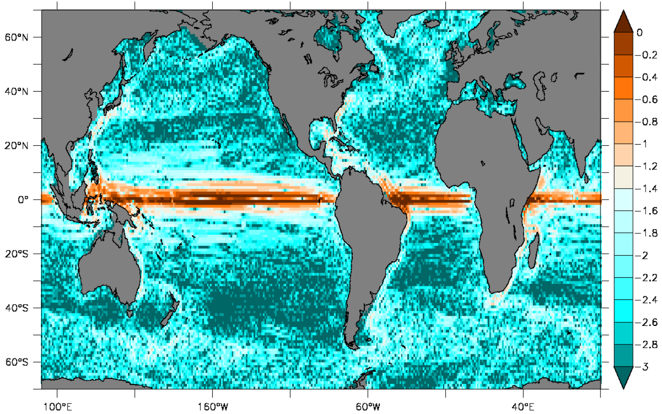
$$\frac{D}{Dt}\left(\boldsymbol{\nabla}^2 h - R^{-2} h\right) + \beta\frac{\partial h}{\partial x} \approx 0$$

with the "Rossby radius" $R = \sqrt{gH}/|f|$ and Earth radius a

- non-linear quasi-geostrophic PV equation: $\partial/\partial t \to D/Dt$ still valid only for $Ro \ll 1$ and $L \ll a$

- *Ro* as $\log_{10}(|\zeta|/|f|)$ at $100\,\mathrm{m}$ in high resolution ocean model



- except for equatorial ocean, $Ro = |\zeta|/|f|$ is well below 0.1

- quasi-geostrophic potential vorticity (PV) equation

$$\frac{D}{Dt}\frac{g}{f}\left(\boldsymbol{\nabla}^2 h - R^{-2}h\right) + \beta v = \frac{D}{Dt}\left[\frac{g}{f_0}\left(\boldsymbol{\nabla}^2 h - R^{-2}h\right) + f_0 + \beta y\right] = 0$$

with material derivative $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$

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with material derivative $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$

- "β-plane" approximation was used at $y' = y_0 + \Delta y$:

$$f(y') = f|_{y_0} + \frac{\partial f}{\partial y}|_{y_0}\Delta y + ... \approx f_0 + \beta\Delta y \equiv f_0 + \beta y$$

with $f_0 = f|_{y_0} = const$ and $\beta = \partial f/\partial y|_{y_0} = const$
it follows that $Df/Dt = D/Dt(f_0 + \beta y) = \boldsymbol{u}\cdot\nabla(\beta y) = \beta v$

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it follows that $Df/Dt = D/Dt(f_0 + \beta y) = \boldsymbol{u}\cdot\boldsymbol{\nabla}(\beta y) = \beta v$

- quasi-geostrophic PV is approximation to full PV for single layer

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$$

- full potential vorticity (PV) equation can be derived from full equations for single layer (see exercises)

- full potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f_0}\left(\boldsymbol{\nabla}^2 h - R^{-2} h\right) + f$$

- full potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f_0}\left(\boldsymbol{\nabla}^2 h - R^{-2} h\right) + f = \zeta - \frac{f_0}{H} h + f$$

with $\zeta \approx (g/f_0)\boldsymbol{\nabla}^2 h$ and $f = f_0 + \beta y$ and $R^2 = gH/f_0^2$

- full potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f_0}\left(\nabla^2 h - R^{-2} h\right) + f = \zeta - \frac{f_0}{H} h + f$$

with $\zeta \approx (g/f_0)\nabla^2 h$ and $f = f_0 + \beta y$ and $R^2 = gH/f_0^2$

- approximate full q

$$\frac{\zeta + f}{h} \quad = \quad \frac{\zeta}{H+\eta} + \frac{f_0}{H+\eta} + \frac{\beta y}{H+\eta}$$

- full potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f_0}\left(\nabla^2 h - R^{-2} h\right) + f = \zeta - \frac{f_0}{H} h + f$$

with $\zeta \approx (g/f_0)\nabla^2 h$ and $f = f_0 + \beta y$ and $R^2 = gH/f_0^2$

- approximate full q using $|\zeta| \ll |f|$ and $|\beta y| \ll |f_0|$ and $|\eta| \ll |H|$

$$\begin{aligned} \frac{\zeta + f}{h} &= \frac{\zeta}{H+\eta} + \frac{f_0}{H+\eta} + \frac{\beta y}{H+\eta} \\ &= \frac{\zeta}{H} + O(Ro) + \frac{f_0/H}{1+\eta/H} + \frac{\beta y}{H} + O(Ro) \end{aligned}$$

- full potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f_0}\left(\nabla^2 h - R^{-2} h\right) + f = \zeta - \frac{f_0}{H} h + f$$

with $\zeta \approx (g/f_0)\nabla^2 h$ and $f = f_0 + \beta y$ and $R^2 = gH/f_0^2$

- approximate full q using $|\zeta| \ll |f|$ and $|\beta y| \ll |f_0|$ and $|\eta| \ll |H|$

$$\begin{aligned}
\frac{\zeta + f}{h} &= \frac{\zeta}{H+\eta} + \frac{f_0}{H+\eta} + \frac{\beta y}{H+\eta} \\
&= \frac{\zeta}{H} + O(Ro) + \frac{f_0/H}{1+\eta/H} + \frac{\beta y}{H} + O(Ro) \\
&= \frac{\zeta}{H} + \frac{f_0}{H}\left(1 - \frac{\eta}{H}\right) + \frac{\beta y}{H} + O(Ro)
\end{aligned}$$

with $1/(1+x) \approx 1 - x$ for small $x = \eta/H$

- full potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f_0}\left(\nabla^2 h - R^{-2} h\right) + f = \zeta - \frac{f_0}{H} h + f$$

with $\zeta \approx (g/f_0)\nabla^2 h$ and $f = f_0 + \beta y$ and $R^2 = gH/f_0^2$

- approximate full q using $|\zeta| \ll |f|$ and $|\beta y| \ll |f_0|$ and $|\eta| \ll |H|$

$$\begin{aligned} \frac{\zeta + f}{h} &= \frac{\zeta}{H+\eta} + \frac{f_0}{H+\eta} + \frac{\beta y}{H+\eta} \\ &= \frac{\zeta}{H} + O(Ro) + \frac{f_0/H}{1+\eta/H} + \frac{\beta y}{H} + O(Ro) \\ &= \frac{\zeta}{H} + \frac{f_0}{H}\left(1 - \frac{\eta}{H}\right) + \frac{\beta y}{H} + O(Ro) \\ &= \left(\zeta + f_0 - (f_0/H)\eta + \beta y\right)/H + O(Ro) \end{aligned}$$

with $1/(1+x) \approx 1 - x$ for small $x = \eta/H$

- full potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f_0}\left(\nabla^2 h - R^{-2} h\right) + f = \zeta - \frac{f_0}{H} h + f$$

with $\zeta \approx (g/f_0)\nabla^2 h$ and $f = f_0 + \beta y$ and $R^2 = gH/f_0^2$

- approximate full q using $|\zeta| \ll |f|$ and $|\beta y| \ll |f_0|$ and $|\eta| \ll |H|$

$$\begin{aligned}
\frac{\zeta + f}{h} &= \frac{\zeta}{H+\eta} + \frac{f_0}{H+\eta} + \frac{\beta y}{H+\eta} \\
&= \frac{\zeta}{H} + O(Ro) + \frac{f_0/H}{1+\eta/H} + \frac{\beta y}{H} + O(Ro) \\
&= \frac{\zeta}{H} + \frac{f_0}{H}\left(1 - \frac{\eta}{H}\right) + \frac{\beta y}{H} + O(Ro) \\
&= \left(\zeta + f_0 - (f_0/H)\eta + \beta y\right)/H + O(Ro) \\
&= \left(\zeta - (f_0/H)h + f\right)/H + f_0/H + O(Ro)
\end{aligned}$$

with $1/(1+x) \approx 1 - x$ for small $x = \eta/H$

- potential vorticity equation for a single layer

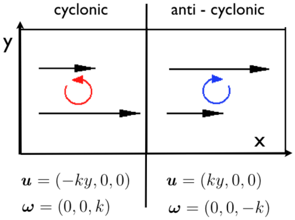
$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q = \zeta - \frac{f_0}{H} h + f$$

q is conserved for fluid parcels in single layer

- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \;\; , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q = \zeta - \frac{f_0}{H} h + f$$

  q is conserved for fluid parcels in single layer
- $h = const$, ζ initially zero, parcel moves northward
  f increases but $q = (f + \zeta)/h$ has to stay constant
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ decreases $\rightarrow$ anticyclonic rotation



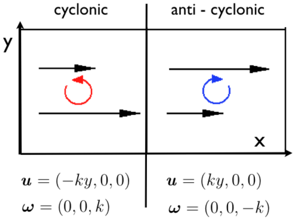
$u = -ay \;,\;\; v = 0 \;\rightarrow\; \zeta = a > 0$: cyclonic (anticlockwise) rotation
$u = +ay \;,\;\; v = 0 \;\rightarrow\; \zeta = a < 0$: anticyclonic (clockwise) rotation

- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q = \zeta - \frac{f_0}{H} h + f$$

  q is conserved for fluid parcels in single layer

- $h = const$, ζ initially zero, parcel moves northward
  f increases but $q = (f + \zeta)/h$ has to stay constant
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ decreases $\rightarrow$ anticyclonic rotation
- $h = const$, ζ initially zero, parcel moves southward
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ increases $\rightarrow$ more cyclonic rotation



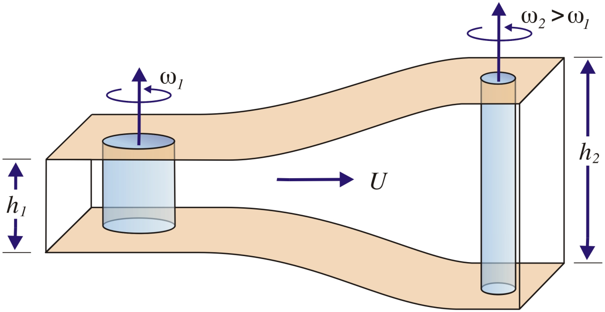
$u = -ay$, $v = 0$ $\rightarrow$ $\zeta = a > 0$: cyclonic (anticlockwise) rotation
$u = +ay$, $v = 0$ $\rightarrow$ $\zeta = a < 0$: anticyclonic (clockwise) rotation

- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q = \zeta - \frac{f_0}{H} h + f$$

  q is conserved for fluid parcels in single layer

- $f = const$, ζ initially zero, parcel moves to deeper water $\rightarrow \zeta = \partial v / \partial x - \partial u / \partial y$ increases $\rightarrow$ cyclonic rotation

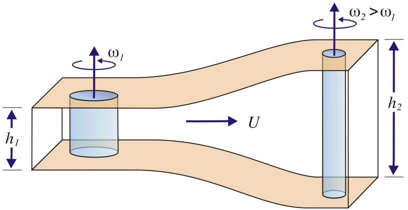


- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f}\left(\nabla^2 h - R^{-2} h\right) + f_0 + \beta y$$

q is (approximately) conserved in single layer for $Ro \ll 1$

- $\zeta = (g/f)\nabla^2 h$ is relative vorticity
- $-(g/f)R^{-2}h$ is stretching vorticity
- $f = f_0 + \beta y$ is planetary vorticity
- h is streamfunction for the quasi-geostrophic flow



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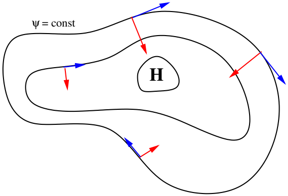
- $\psi = gh/f_0$ is streamfunction for the quasi-geostrophic flow

$$u \approx -\frac{g}{f_0}\frac{\partial h}{\partial y} = -\frac{\partial \psi}{\partial y} \quad , \quad v \approx \frac{g}{f_0}\frac{\partial h}{\partial x} = \frac{\partial \psi}{\partial x}$$



$$\boldsymbol{u} = \begin{pmatrix} -\partial\psi/\partial y \\ \partial\psi/\partial x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ 0 \end{pmatrix} = \boldsymbol{k} \times \nabla\psi$$

- $\boldsymbol{u}$ (blue arrow): anti-clockwise rotation of $\nabla\psi$ (red arrow) by 90^o

Recapitulation

- Layered models
- Gravity waves without rotation
- Gravity waves with rotation

Waves

- Kelvin waves
- Quasi-geostrophic approximation
- Potential vorticity
- Geostrophic adjustment

- consider the (linearized) layered model with $f = const$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}\ , \quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}\ , \quad \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

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- (linearized, $D/Dt \to \partial/\partial t$) potential vorticity equation

$$\frac{\partial q}{\partial t} = 0\ \ ,\quad q = \frac{\zeta + f}{h} \approx (\zeta - \frac{f}{H}h + f)/H$$

- f in q for $f = const$ does not matter $\to q = \zeta - (f/H)h$

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- consider as initial condition $\boldsymbol{u} = 0$ and h a step function such that

$$h|_{t=0} = \begin{cases} h_0, & \text{if } x < 0 \\ -h_0, & \text{if } x > 0 \end{cases} \quad \to \quad q_0 = q|_{t=0} = \begin{cases} -fh_0/H, & \text{if } x < 0 \\ fh_0/H, & \text{if } x > 0 \end{cases}$$

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- using $q(t) = q_0$ steady state solution ($t \to \infty$) is given by

$$fv_\infty = g\frac{\partial h_\infty}{\partial x}\ ,\quad fu_\infty = -g\frac{\partial h_\infty}{\partial y}$$

$$\to\ q_\infty = \frac{g}{f}\frac{\partial^2 h_\infty}{\partial x^2} + \frac{g}{f}\frac{\partial^2 h_\infty}{\partial y^2} - \frac{f}{H}h_\infty = q_0$$

$\to \boldsymbol{\nabla}^2 h_\infty - R^{-2}h_\infty = (f/g)q_0$ with Rossby radius $R = \sqrt{gH}/|f|$

- steady state solution ($t \to \infty$) is given by

$$\nabla^2 h_\infty - R^{-2} h_\infty = (f/g) q_0 = \begin{cases} -R^{-2} h_0, & \text{if } x < 0 \\ R^{-2} h_0, & \text{if } x > 0 \end{cases}$$

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- solution of h_∞ is given by

$$h(x)_\infty = \begin{cases} h_0(1 - e^{x/R}), & \text{if } x < 0 \\ -h_0(1 - e^{-x/R}), & \text{if } x > 0 \end{cases}$$

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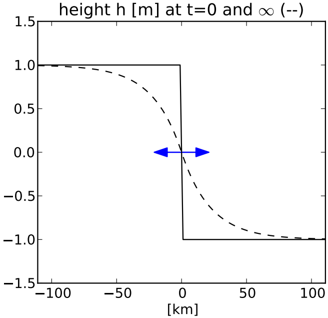
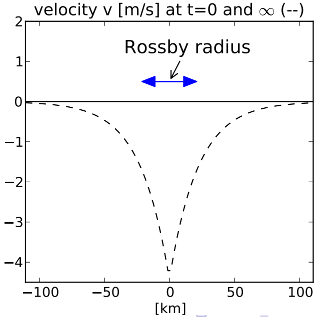
- initial and steady state solution of h are given by

$$h|_{t=0} = \begin{cases} h_0, & \text{if } x < 0 \\ -h_0, & \text{if } x > 0 \end{cases}, \quad h|_\infty = \begin{cases} h_0(1 - e^{x/R}), & \text{if } x < 0 \\ -h_0(1 - e^{-x/R}), & \text{if } x > 0 \end{cases}$$

with Rossby radius $R = \sqrt{gH}/|f|$





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- velocities from $fv_\infty = g\partial h_\infty/\partial x$ and $fu_\infty = -g\partial h_\infty/\partial y$

$$u_\infty = 0 \quad , \quad v_\infty = (g/f) \begin{cases} -h_0/Re^{x/R}, & \text{if } x < 0 \\ -h_0/Re^{-x/R}, & \text{if } x > 0 \end{cases} = -\frac{gh_0}{fR} e^{-|x|/R}$$