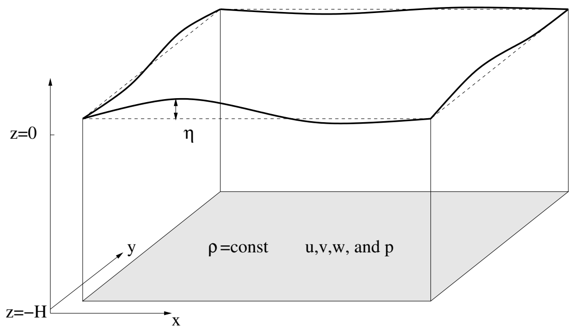
11 – Waves and Instabilities

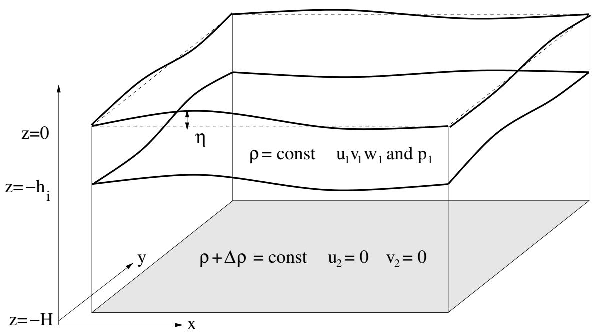
- "barotropic model" and "baroclinic model"

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \to g\Delta\rho/\rho_0$ ("baroclinic")





- consider the layered model (first without $\boldsymbol{u} \cdot \boldsymbol{\nabla u}$ for simplicity)

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- take curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{\partial}{\partial x}(2.\text{eqn}) : \frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}(fu) = -g\frac{\partial}{\partial x}\frac{\partial h}{\partial y}$$

$$\frac{\partial}{\partial y}(1.\text{eqn}) : \frac{\partial}{\partial t}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}(fv) = -g\frac{\partial}{\partial y}\frac{\partial h}{\partial x}$$

subtract both

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(fu) + \frac{\partial}{\partial y}(fv) = 0$$

$$\frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and with $\beta = \partial f/\partial y$

- assume small Rossby number Ro, i.e. dominant geostrophic balance

$$O(Ro) - fv = -g\frac{\partial h}{\partial x} \quad , \quad O(Ro) + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- then $v \approx (g/f)\,\partial h/\partial x$ and $u \approx -(g/f)\,\partial h/\partial y$
- relative vorticity ζ becomes

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \approx \frac{\partial}{\partial x}\left(\frac{g}{f}\frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{g}{f}\frac{\partial h}{\partial y}\right)$$

$$= \frac{g}{f}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) - \frac{g}{f^2}\frac{\partial h}{\partial y}\frac{\partial f}{\partial y} \approx \frac{g}{f}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)$$

assuming $|(g\beta/f^2)\,\partial h/\partial y| \ll |\partial v/\partial x|$, i.e small variations of f

$$\frac{g}{f^2}\frac{\partial h}{\partial y}\frac{\partial f}{\partial y} \sim \frac{U}{\Omega}\frac{\Omega}{a} , \quad \frac{\partial v}{\partial x} \sim \frac{U}{L} \quad \rightarrow \quad \frac{U}{a} \ll \frac{U}{L} \text{ if } L \ll a$$

with Earth radius a , i.e. only length scales L smaller than a are valid

- assume small Rossby number Ro, i.e. dominant geostrophic balance

$$O(Ro) - fv = -g\frac{\partial h}{\partial x} \quad , \quad O(Ro) + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- curl $\zeta \approx (g/f)\,(\partial^2 h/\partial x^2 + \partial^2 h/\partial y^2)$ for $Ro \ll 1$ and $L \ll a$

$$\begin{aligned}\frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v &= 0 \\ \frac{g}{f}\frac{\partial}{\partial t}\nabla^2 h - \frac{f}{H}\frac{\partial h}{\partial t} + \beta\frac{g}{f}\frac{\partial h}{\partial x} &\approx 0 \\ \frac{\partial}{\partial t}\left(\nabla^2 h - R^{-2}h\right) + \beta\frac{\partial h}{\partial x} &\approx 0\end{aligned}$$

with the "Rossby radius" $R = \sqrt{gH}/|f|$ and Earth radius a

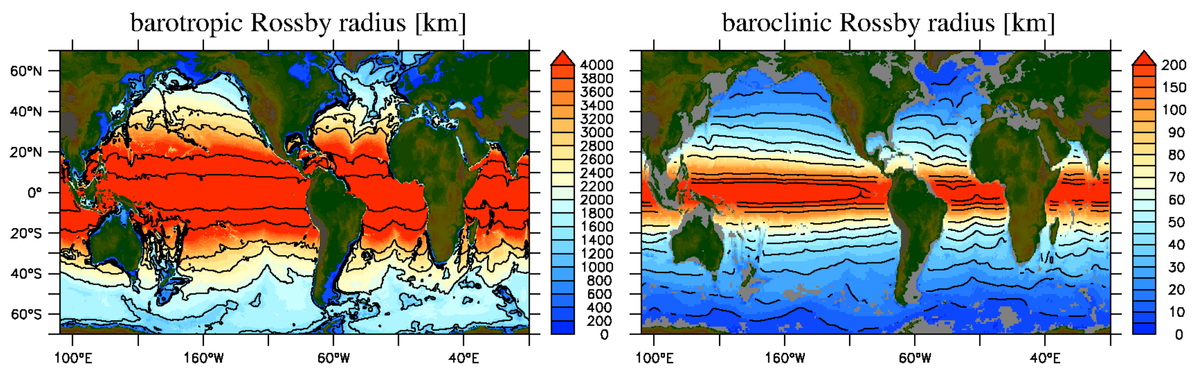
- single equation in h: quasi-geostrophic potential vorticity equation valid for $Ro \ll 1$ and $L \ll a$

- quasi-geostrophic potential vorticity (PV) equation

$$\frac{\partial}{\partial t}\left(\nabla^2 h - R^{-2}h\right) + \beta\frac{\partial h}{\partial x} = 0$$

valid for $Ro \ll 1$ and $L \ll a$
with the "Rossby radius" $R = \sqrt{gH}/|f|$ and Earth radius a

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \to g\Delta\rho/\rho_0$ ("baroclinic")



- consider again the layered model, but this time with $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

- take curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\begin{aligned} \frac{\partial}{\partial x}(2.\text{eqn}) \ : \ \frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}(\boldsymbol{u} \cdot \boldsymbol{\nabla} v) + \frac{\partial}{\partial x}(fu) &= -g\frac{\partial}{\partial x}\frac{\partial h}{\partial y} \\ \frac{\partial}{\partial y}(1.\text{eqn}) \ : \ \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}(\boldsymbol{u} \cdot \boldsymbol{\nabla} u) - \frac{\partial}{\partial y}(fv) &= -g\frac{\partial}{\partial y}\frac{\partial h}{\partial x} \end{aligned}$$

subtract both

$$\begin{aligned} \frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + \frac{\partial}{\partial x}(fu) + \frac{\partial}{\partial y}(fv) &= 0 \\ \frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v &= 0 \end{aligned}$$

with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and with $\beta = \partial f/\partial y$

- curl of momentum equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

- calculate

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ &= \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \end{aligned}$$

- now use $|\zeta| \ll |f| \to U/L \ll \Omega \to U/(\Omega L) = Ro \ll 1$

$$\frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v \approx 0$$

for $Ro \ll 1$

- assume again $Ro \ll 1$, i.e. dominant geostrophic balance

$$O(Ro) - fv = -g\frac{\partial h}{\partial x} \quad , \quad O(Ro) + fu = -g\frac{\partial h}{\partial y}$$
$$\frac{Dh}{Dt} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

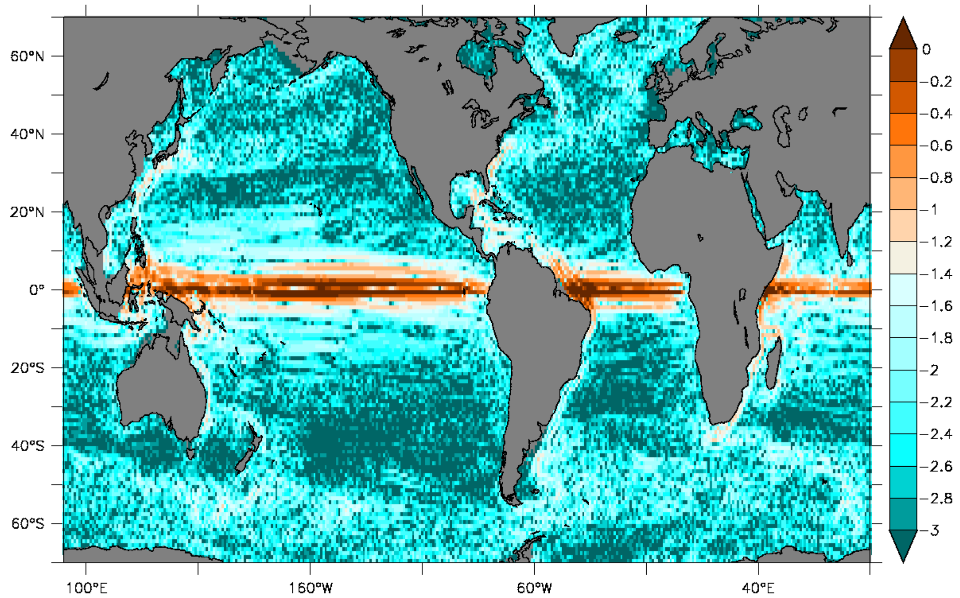
- curl $\zeta \approx (g/f)\,(\partial^2 h/\partial x^2 + \partial^2 h/\partial y^2)$ for $Ro \ll 1$ and $L \ll a$

$$\frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v \approx 0$$
$$\frac{D}{Dt}\left(\frac{g}{f}\nabla^2 h\right) - \frac{f}{H}\frac{Dh}{Dt} + \beta\frac{g}{f}\frac{\partial h}{\partial x} \approx 0$$
$$\frac{D}{Dt}\left(\nabla^2 h - R^{-2}h\right) + \beta\frac{\partial h}{\partial x} \approx 0$$

with the "Rossby radius" $R = \sqrt{gH}/|f|$ and Earth radius a

- non-linear quasi-geostrophic PV equation: $\partial/\partial t \to D/Dt$ still valid only for $Ro \ll 1$ and $L \ll a$

- Ro as $\log_{10}(|\zeta|/|f|)$ at $100\,\mathrm{m}$ in high resolution ocean model



- except for equatorial ocean, $Ro = |\zeta|/|f|$ is well below 0.1

- quasi-geostrophic potential vorticity (PV) equation

$$\frac{D}{Dt}\frac{g}{f}\left(\nabla^2 h - R^{-2}h\right) + \beta v = \frac{D}{Dt}\left[\frac{g}{f_0}\left(\nabla^2 h - R^{-2}h\right) + f_0 + \beta y\right] = 0$$

with material derivative $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$

- "β-plane" approximation was used at $y' = y_0 + \Delta y$:

$$f(y') = f|_{y_0} + \frac{\partial f}{\partial y}|_{y_0}\Delta y + ... \approx f_0 + \beta\Delta y \equiv f_0 + \beta y$$

with $f_0 = f|_{y_0} = const$ and $\beta = \partial f/\partial y|_{y_0} = const$
it follows that $Df/Dt = D/Dt(f_0 + \beta y) = \boldsymbol{u}\cdot\nabla(\beta y) = \beta v$

- quasi-geostrophic PV is approximation to full PV for single layer

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$$

- full potential vorticity (PV) equation can be derived from full equations for single layer (see exercises)

- full potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f_0}\left(\nabla^2 h - R^{-2}h\right) + f = \zeta - \frac{f_0}{H}h + f$$

with $\zeta \approx (g/f_0)\nabla^2 h$ and $f = f_0 + \beta y$ and $R^2 = gH/f_0^2$

- approximate full q using $|\zeta| \ll |f|$ and $|\beta y| \ll |f_0|$ and $|\eta| \ll |H|$

$$\begin{aligned}
\frac{\zeta + f}{h} &= \frac{\zeta}{H+\eta} + \frac{f_0}{H+\eta} + \frac{\beta y}{H+\eta} \\
&= \frac{\zeta}{H} + O(Ro) + \frac{f_0/H}{1+\eta/H} + \frac{\beta y}{H} + O(Ro) \\
&= \frac{\zeta}{H} + \frac{f_0}{H}\left(1 - \frac{\eta}{H}\right) + \frac{\beta y}{H} + O(Ro) \\
&= \left(\zeta + f_0 - (f_0/H)\eta + \beta y\right)/H + O(Ro) \\
&= \left(\zeta - (f_0/H)h + f\right)/H + f_0/H + O(Ro)
\end{aligned}$$

with $1/(1+x) \approx 1 - x$ for small $x = \eta/H$

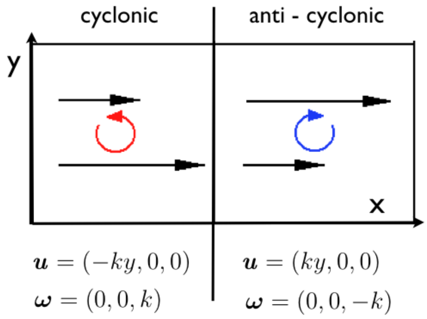
- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q = \zeta - \frac{f_0}{H}h + f$$

  q is conserved for fluid parcels in single layer

- $h = const$, ζ initially zero, parcel moves northward
  f increases but $q = (f + \zeta)/h$ has to stay constant
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ decreases $\rightarrow$ anticyclonic rotation

- $h = const$, ζ initially zero, parcel moves southward
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ increases $\rightarrow$ more cyclonic rotation



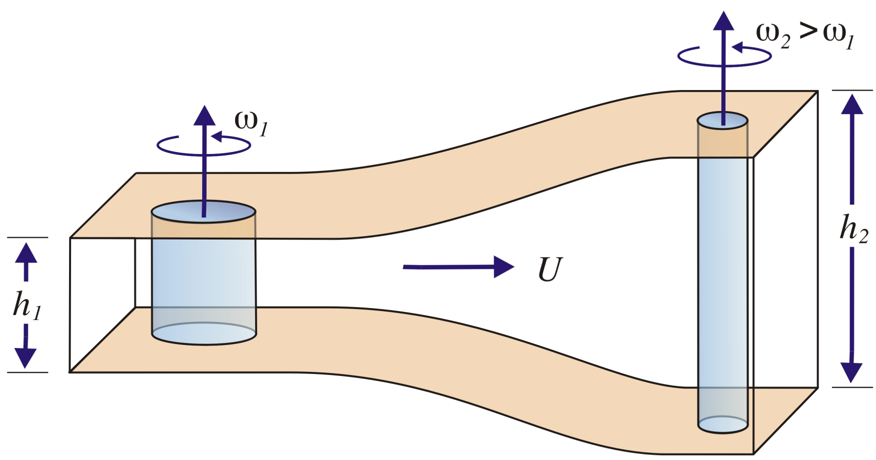
$u = -ay$, $v = 0$ $\rightarrow$ $\zeta = a > 0$: cyclonic (anticlockwise) rotation
$u = +ay$, $v = 0$ $\rightarrow$ $\zeta = a < 0$: anticyclonic (clockwise) rotation

- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q = \zeta - \frac{f_0}{H}h + f$$

  q is conserved for fluid parcels in single layer

- $f = const$, ζ initially zero, parcel moves to deeper water
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ increases $\rightarrow$ cyclonic rotation

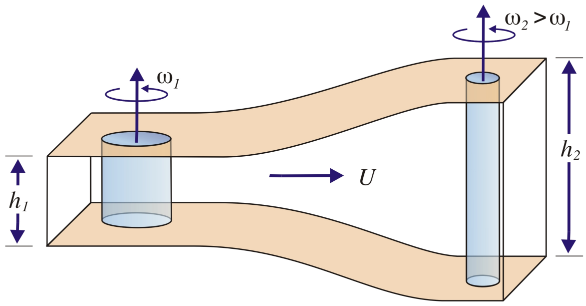


- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f}\left(\nabla^2 h - R^{-2} h\right) + f_0 + \beta y$$

q is (approximately) conserved in single layer for $Ro \ll 1$

- $\zeta = (g/f)\nabla^2 h$ is relative vorticity
- $-(g/f)R^{-2}h$ is stretching vorticity
- $f = f_0 + \beta y$ is planetary vorticity
- h is streamfunction for the quasi-geostrophic flow



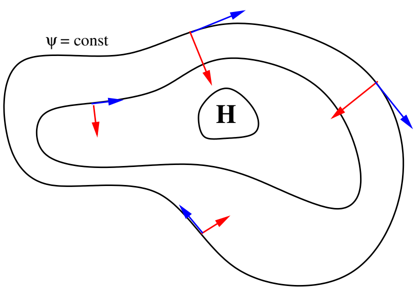
- quasi-geostrophic potential vorticity equation

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{g}{f}\left(\nabla^2 h - R^{-2} h\right) + f_0 + \beta y$$

q is (approximately) conserved in single layer for $Ro \ll 1$

- $\psi = gh/f_0$ is streamfunction for the quasi-geostrophic flow

$$u \approx -\frac{g}{f_0}\frac{\partial h}{\partial y} = -\frac{\partial \psi}{\partial y} \quad , \quad v \approx \frac{g}{f_0}\frac{\partial h}{\partial x} = \frac{\partial \psi}{\partial x}$$



$$\boldsymbol{u} = \begin{pmatrix} -\partial\psi/\partial y \\ \partial\psi/\partial x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ 0 \end{pmatrix} = \boldsymbol{k} \times \nabla\psi$$

- $\boldsymbol{u}$ (blue arrow): anti-clockwise rotation of $\nabla\psi$ (red arrow) by 90^o