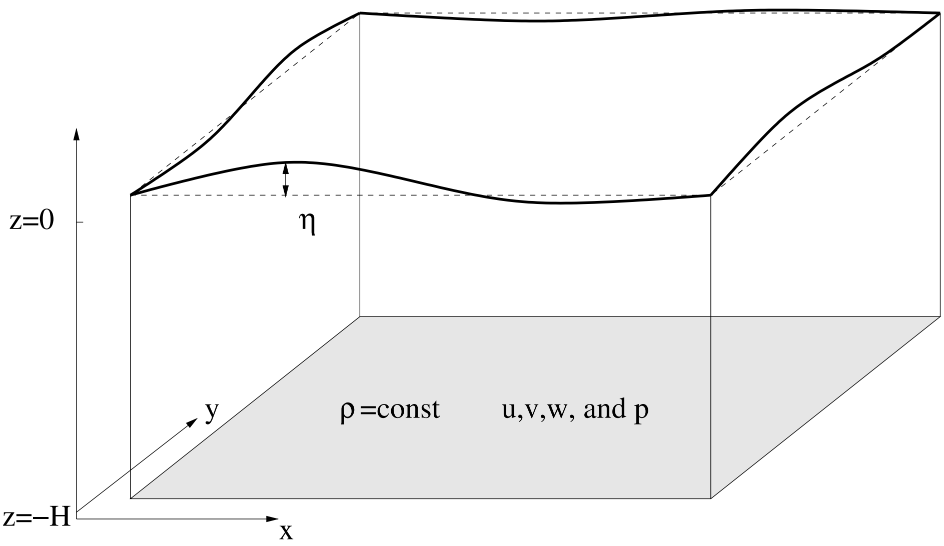
11 – Waves and Instabilities

- consider a single layer system in hydrostatic approximation
- assume $\rho = const$ and no vertical shear $\partial u/\partial z = \partial v/\partial z = 0$



- with sea level at $z = \eta$ and the bottom at $z = -H$

- consider a single layer system in hydrostatic approximation

$$\begin{aligned}
\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} u - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} v + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\frac{\partial p}{\partial z} &= -g\rho \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}$$

- assume $\rho = const$ and no vertical shear $\partial u/\partial z = \partial v/\partial z = 0$
- now vertically integrate continuity equation from bottom to top

$$\begin{aligned}
\int_{-H}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz + \int_{-H}^{\eta}\frac{\partial w}{\partial z}dz &= 0 \\
(H+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w|_{\eta} - w|_{-H} &= 0
\end{aligned}$$

with sea level at $z = \eta$ and the bottom at $z = -H$

- assume $\rho = const$ and no vertical shear $\partial u/\partial z = \partial v/\partial z = 0$
- vertically integrate continuity equation from bottom to top

$$(H+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w|_{\eta} - w|_{-H} = 0$$

- now use kinematic boundary conditions

$$w_{-H} = 0 \quad , \quad w|_{\eta} = \frac{\partial \eta}{\partial t} + u|_{\eta}\frac{\partial \eta}{\partial x} + v|_{\eta}\frac{\partial \eta}{\partial y}$$

which means no mass flux through upper and lower boundaries

- this yields

$$\begin{aligned}
(H+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} &= 0 \\
h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} &= 0 \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) &= 0
\end{aligned}$$

which becomes a layer thickness equation for $h = H + \eta$

- assume $\rho = const$ and integrate hydrostatic balance from z to top

$$\begin{aligned}\frac{\partial p}{\partial z} &= -g\rho \\ \int_z^\eta \frac{\partial p}{\partial z} dz &= p|_\eta - p|_z = -g\rho \int_z^\eta dz = -g\rho(\eta - z) \\ p &= p|_\eta - g\rho(z-\eta) \\ \nabla p &= g\rho\nabla\eta = g\rho\nabla h\end{aligned}$$

with layer thickness $h = \eta + H$

- momentum equation becomes

$$\begin{aligned}\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\nabla u - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\nabla v + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial h}{\partial y}\end{aligned}$$

since $h(x,y,t)$ and $\partial u/\partial z = \partial v/\partial z = 0$ equations are now 2-D

- single layer system in hydrostatic approximation

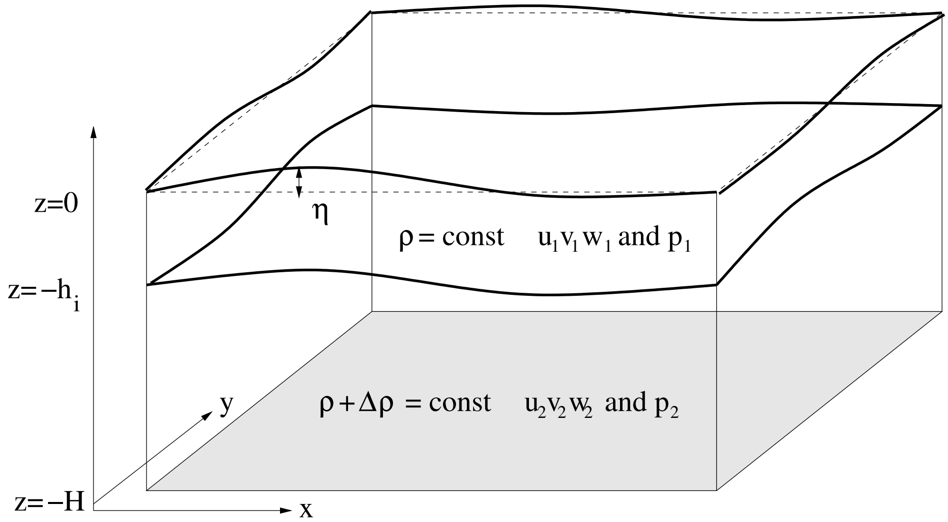
$$\begin{aligned}\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\nabla u - fv &= -g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\nabla v + fu &= -g\frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) &= 0\end{aligned}$$

- neglecting momentum advection for simplicity
  and assuming $H \gg \eta$ in $h = H + \eta \rightarrow \nabla\cdot(\boldsymbol{u}h) \approx H\nabla\cdot\boldsymbol{u}$

$$\begin{aligned}\frac{\partial u}{\partial t} - fv &= -g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0\end{aligned}$$

simple system which contains almost all relevant dynamics

- two layers with $\rho_1 = \rho = const$ and $\rho_2 = \rho + \Delta\rho = const$
- sea surface at $z = \eta$ and layer interface $z = -h_i$
- assume again no vertical shear $\partial u_{1,2}/\partial z = \partial v_{1,2}/\partial z = 0$ in layers



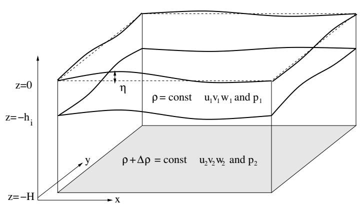
- pressure gradient in upper layer $g\rho\boldsymbol{\nabla}\eta$
- pressure gradient in lower layer $-g(\rho+\Delta\rho)\boldsymbol{\nabla} h_i + g\rho\boldsymbol{\nabla}(\eta + h_i)$

- upper layer equations

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} u_1 - f v_1 &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v_1}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} v_1 + f u_1 &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial}{\partial t}(\eta + h_i) + \frac{\partial}{\partial x}u_1(\eta + h_i) + \frac{\partial}{\partial y}v_1(\eta + h_i) &= 0
\end{aligned}
$$

- lower layer equations

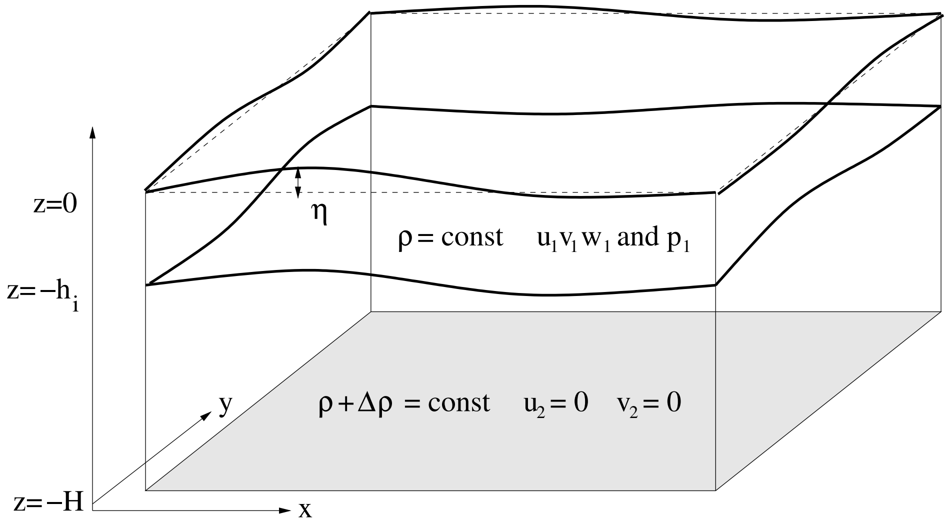
$$
\begin{aligned}
\frac{\partial u_2}{\partial t} + \boldsymbol{u}_2 \cdot \boldsymbol{\nabla} u_2 - f v_2 &= g\frac{\Delta\rho}{\rho}\frac{\partial h_i}{\partial x} - g\frac{\partial \eta}{\partial x} \\
\frac{\partial v_2}{\partial t} + \boldsymbol{u}_2 \cdot \boldsymbol{\nabla} v_2 + f u_2 &= g\frac{\Delta\rho}{\rho}\frac{\partial h_i}{\partial y} - g\frac{\partial \eta}{\partial y} \\
\frac{\partial}{\partial t}(H - h_i) + \frac{\partial}{\partial x}u_2(H - h_i) + \frac{\partial}{\partial y}v_2(H - h_i) &= 0
\end{aligned}
$$



- assume that lower layer is infinitively deep and motionless

$$0 = g\frac{\Delta\rho}{\rho}\boldsymbol{\nabla} h_i - g\boldsymbol{\nabla}\eta \quad \rightarrow \quad \frac{\Delta\rho}{\rho}h_i - \eta = const = 0 \quad \rightarrow \quad \eta = \frac{\Delta\rho}{\rho}h_i$$

vanishing pressure variations in lower layer



- assume that lower layer is infinitively deep and motionless

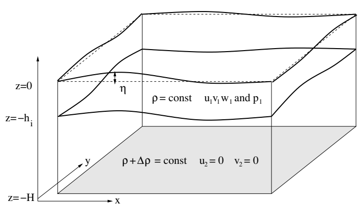
$$0 = g\frac{\Delta\rho}{\rho}\boldsymbol{\nabla} h_i - g\boldsymbol{\nabla}\eta \quad \rightarrow \quad \eta = \frac{\Delta\rho}{\rho}h_i$$

vanishing pressure variations in lower layer

- upper layer equations become

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} u_1 - f v_1 &= -g'\frac{\partial h_i}{\partial x} \\
\frac{\partial v_1}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} v_1 + f u_1 &= -g'\frac{\partial h_i}{\partial y} \\
\frac{\partial}{\partial t}h_i + \frac{\partial}{\partial x}(u_1 h_i) + \frac{\partial}{\partial y}(v_1 h_i) &= 0
\end{aligned}$$

with "reduced gravity" $g' = g\Delta\rho/\rho$

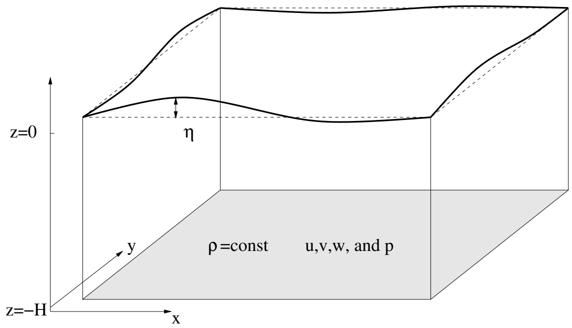


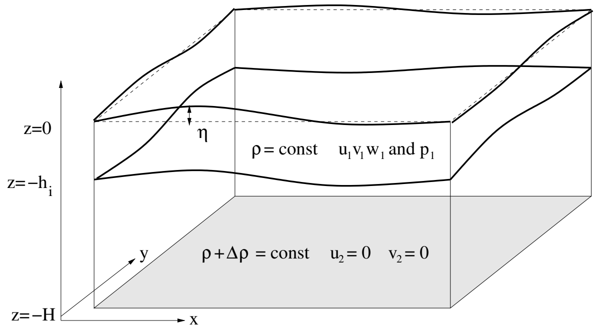
- "barotropic model" and "baroclinic model"

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \to g\Delta\rho/\rho_0$ ("baroclinic")





- consider the (linearized) layered model with $f = 0$
  and also set y dependency to zero $\to$ $v = 0$

$$\frac{\partial u}{\partial t} - \cancel{fv} = -g\frac{\partial h}{\partial x} \; , \; \cancel{\frac{\partial v}{\partial t}} + \cancel{fu} = \cancel{-g\frac{\partial h}{\partial y}} \; , \; \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}\right) = 0$$

- combine momentum and thickness equation to wave equation

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial t} = -g\frac{\partial}{\partial x}\frac{\partial h}{\partial x} \; , \; \frac{\partial}{\partial t}\frac{\partial h}{\partial t} + H\frac{\partial}{\partial t}\frac{\partial u}{\partial x} = 0 \;\to\; \frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x^2} = 0$$

- try particular solution $h(x,t) = \sin k(x - ct)$

$$\frac{\partial h}{\partial t} = -kc\cos k(x-ct) \quad , \quad \frac{\partial^2 h}{\partial t^2} = -(kc)^2 \sin k(x-ct)$$

$$\frac{\partial h}{\partial x} = k\cos k(x-ct) \quad , \quad \frac{\partial^2 h}{\partial x^2} = -k^2 \sin k(x-ct)$$

- this works as long as

  $$-(kc)^2\sin(..) + k^2 gH\sin(..) = 0 \;\to\; c^2 = gH \;\to\; c = \pm\sqrt{gH}$$

  which is the dispersion relation for a gravity wave (for $f = 0$)

- gravity wave equation (for $f = 0$)

$$\frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x^2} = 0$$

- a particular solution is $h(x,t) = \sin k(x - ct)$
- $h = A \sin k(x - ct)$ with constant amplitude A is also solution and also $h = A\sin(k(x - ct) + \phi)$ with constant phase ϕ
- more general wave solution is

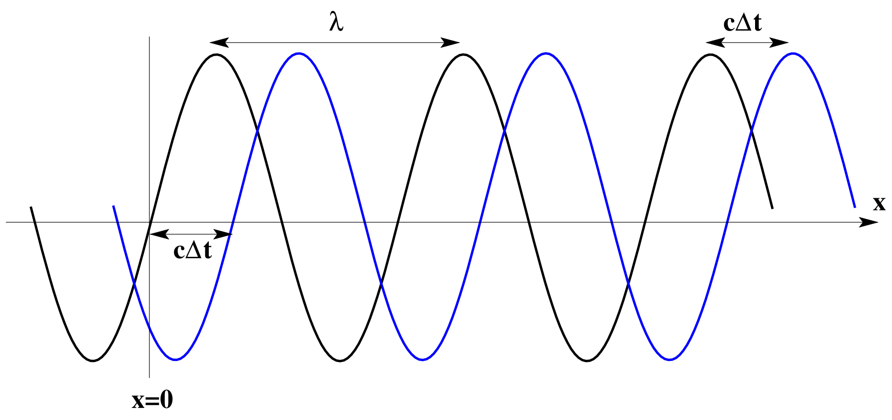
$$h = A \sin k(x - ct) + B \cos k(x - ct)$$

- or write more compact as

$$\begin{aligned} h &= \text{Re}\left\{Ae^{ik(x-ct)}\right\} = \text{Re}\left\{(A_r + iA_i)(\cos k(x - ct) + i \sin k(x - ct))\right\} \\ &= \text{Re}\left\{A_r \cos k(x - ct) + iA_r \sin k(x - ct)\right\} \\ &\quad +\text{Re}\left\{iA_i \cos k(x - ct) - A_i \sin k(x - ct)\right\} \\ &= A_r \cos k(x - ct) - A_i \sin k(x - ct) \end{aligned}$$

with complex constant A with $\text{Re}\{A\} = A_r$ and $\text{Im}\{A\} = A_i$
with Euler relation $e^{i\phi} = \cos\phi + i \sin\phi$

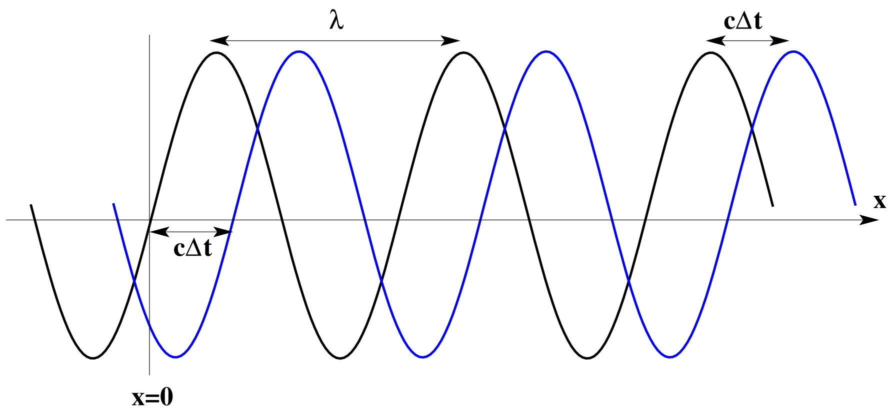
- gravity wave equation (for $f = 0$) $\partial^2 h/\partial t^2 - gH\partial^2 h/\partial x^2 = 0$
- wave solution is given by $h = Ae^{ik(x-ct)}$ with complex amplitude A (Re is often dropped for convenience) as long as $c = \pm\sqrt{gH}$
- consider $h = \sin k(x - ct)$ at $t = 0 \rightarrow h = \sin kx$ (black line) $\rightarrow$ wavelength is $\lambda = 2\pi/k$, k is wavenumber
- consider h at $t = 0$ (black line) and at later time $t = \Delta t$ (blue line) phase where $h = 0$ was at $t = 0$ at $x = 0$ but at $t = \Delta t$ at $x = c\Delta t$ $\rightarrow$ $c = dx/dt$ is the velocity at which constant phase propagates $\rightarrow$ phase velocity



- gravity wave equation (for $f = 0$) $\partial^2 h/\partial t^2 - gH\partial^2 h/\partial x^2 = 0$
- wave solution is given by $h = Ae^{ik(x-ct)}$ with complex amplitude A (Re is often dropped for convenience) as long as $c = \pm\sqrt{gH}$
- wavelength $\lambda = 2\pi/k$ with wavenumber k
- phase velocity c with dispersion relation $c = \pm\sqrt{gH}$
- rewrite solution as $h = Ae^{i(kx-\omega t)}$ with frequency $\omega = ck$ and

$$\omega = \pm k\sqrt{gH}$$

- $T = 2\pi/\omega$ is the period in which a fixed phase pass a fixed point



- consider the (linearized) layered model with $f = 0$
  but now include y dependency $\rightarrow$ plane wave

$$\frac{\partial u}{\partial t} - \cancel{fv} = -g\frac{\partial h}{\partial x}\ ,\ \frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial h}{\partial y}\ ,\ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- combine momentum and thickness equation to wave equation

$$\nabla\cdot\frac{\partial \mathbf{u}}{\partial t} = -\nabla\cdot g\nabla h\ ,\ \frac{\partial}{\partial t}\frac{\partial h}{\partial t} + \frac{\partial}{\partial t}H\nabla\cdot\mathbf{u} = 0 \ \rightarrow\ \frac{\partial^2 h}{\partial t^2} - gH\nabla^2 h = 0$$

- wave solution $h = A\exp i(k_1 x + k_2 y - \omega t) = A\exp i(\mathbf{k}\cdot\mathbf{x} - \omega t)$

$$\frac{\partial h}{\partial t} = -i\omega A\exp i(\ldots)\quad ,\quad \frac{\partial^2 h}{\partial t^2} = (i\omega)^2 A\exp i(\ldots) = -\omega^2 A\exp i(\ldots)$$

$$\nabla h = i\mathbf{k}A\exp i(\ldots)\quad ,\quad \nabla\cdot\nabla h = i^2\mathbf{k}\cdot\mathbf{k}A\exp i(\ldots) = -k^2 A\exp i(\ldots)$$

  with wavenumber vector $\mathbf{k} = (k_1, k_2)$ and $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$
- this works as long as

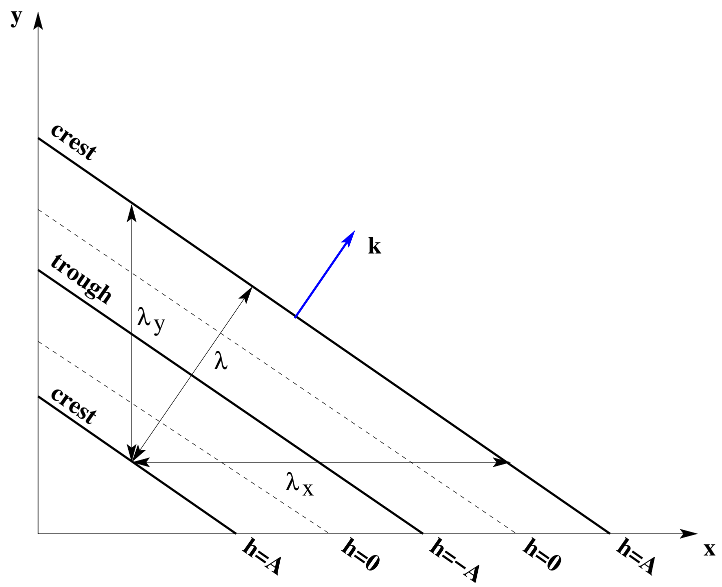
$$-\omega^2\exp i(..) + k^2 gH\exp i(..) = 0\ \rightarrow\ \omega^2 = k^2 gH\ \rightarrow\ \omega = \pm k\sqrt{gH}$$

  which is still the dispersion relation for a gravity wave (for $f = 0$)

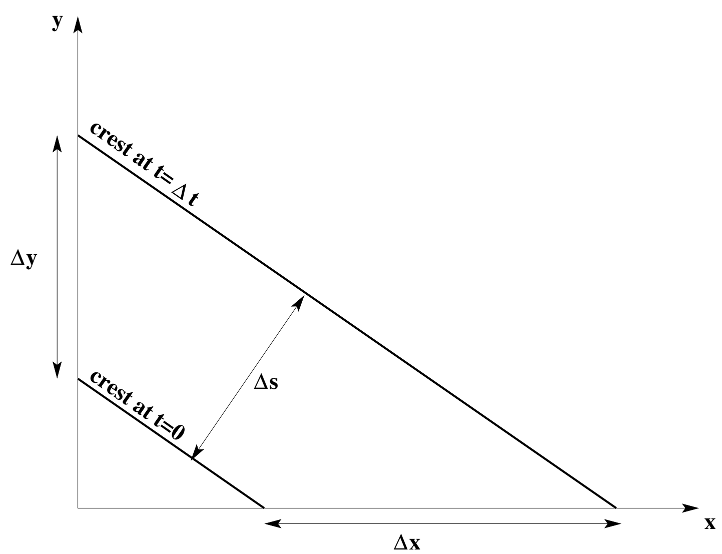
- plane gravity wave (for $f = 0$) is given by $h = A \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$
- wavenumber vector $\boldsymbol{k}$ gives direction of phase propagation
- zonal and meridional wave length λ_x, λ_y and "real" wavelength λ

$$\lambda_x = 2\pi/k_1 \;,\; \lambda_y = 2\pi/k_2 \;,\; \lambda = 2\pi/k = 2\pi/\sqrt{k_1^2 + k_2^2}$$

but note that $\lambda \neq \sqrt{\lambda_x^2 + \lambda_y^2}$



- plane gravity wave (for $f = 0$) is given by $h = A \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$
- phase velocity c with dispersion relation $c = \pm\sqrt{gH}$ or $\omega = \pm k\sqrt{gH}$
- phase propagates from $t = 0$ to $t = \Delta t$ the distance $\Delta s = c\Delta t$
- along x-axis the distance $\Delta x = c_x \Delta t = \Delta t\, \omega/k_1 \rightarrow c_x = \omega/k_1$
  along y-axis the distance $\Delta y = c_y \Delta t = \Delta t\, \omega/k_2 \rightarrow c_y = \omega/k_2$
  but note that $c \neq \sqrt{c_x^2 + c_y^2}$



- add two waves with different $\boldsymbol{k}$ and ω but same amplitude

$$
\begin{aligned}
h &= A\cos(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)+A\cos(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega' t)\\
&= 2A\cos\left(\frac{\boldsymbol{k}'-\boldsymbol{k}}{2}\cdot\boldsymbol{x}-\frac{\omega'-\omega}{2}t\right)\cos\left(\frac{\boldsymbol{k}'+\boldsymbol{k}}{2}\cdot\boldsymbol{x}-\frac{\omega'+\omega}{2}t\right)
\end{aligned}
$$

with $\omega=|\boldsymbol{k}|\sqrt{gH}=\omega(\boldsymbol{k})$ and $\omega'=|\boldsymbol{k}'|\sqrt{gH}=\omega(\boldsymbol{k}')$

- for similar wave numbers $\boldsymbol{k}'=\boldsymbol{k}+\Delta\boldsymbol{k}$ with small $\Delta\boldsymbol{k}$

$$
\begin{aligned}
\omega(\boldsymbol{k}') &= \omega(\boldsymbol{k}+\Delta\boldsymbol{k})=\omega(\boldsymbol{k})+\frac{\partial\omega}{\partial k_1}\Delta k_x+\frac{\partial\omega}{\partial k_2}\Delta k_y+\cdots\\
&= \omega(\boldsymbol{k})+\boldsymbol{c}_g\cdot\Delta\boldsymbol{k}+\cdots \quad\rightarrow\quad \omega(\boldsymbol{k}')-\omega(\boldsymbol{k})\approx\boldsymbol{c}_g\cdot\Delta\boldsymbol{k}
\end{aligned}
$$

with the *group velocity* $\boldsymbol{c}_g=\left(\frac{\partial\omega}{\partial k_1},\frac{\partial\omega}{\partial k_2}\right)=\partial\omega/\partial\boldsymbol{k}$

$$
\begin{aligned}
h &\approx 2A\cos\left(\frac{\Delta\boldsymbol{k}}{2}\cdot\boldsymbol{x}-\frac{\boldsymbol{c}_g\cdot\Delta\boldsymbol{k}}{2}t\right)\cos\left(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t\right)\\
h &\approx 2A\cos\left(\frac{\Delta\boldsymbol{k}}{2}\cdot[\boldsymbol{x}-\boldsymbol{c}_g t]\right)\cos\left(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t\right)
\end{aligned}
$$

- amplitude modulation with speed $\boldsymbol{c}_g$ and wave length $\Delta\boldsymbol{k}$

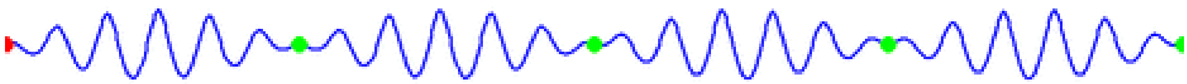
- add two waves with different $\boldsymbol{k}$ and ω but same amplitude

$$
\begin{aligned}
h &= A\cos(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)+A\cos(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega' t)\\
h &\approx 2A\cos\left(\frac{\Delta\boldsymbol{k}}{2}\cdot[\boldsymbol{x}-\boldsymbol{c}_g t]\right)\cos\left(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t\right)
\end{aligned}
$$

with the wavenumber difference $\Delta\boldsymbol{k}=\boldsymbol{k}'-\boldsymbol{k}$

and the *group velocity* $\boldsymbol{c}_g=\left(\frac{\partial\omega}{\partial k_1},\frac{\partial\omega}{\partial k_2}\right)=\partial\omega/\partial\boldsymbol{k}$

- amplitude modulation with speed $\boldsymbol{c}_g$ and wave length $\Delta\boldsymbol{k}$



- $\boldsymbol{c}_g$ is the speed at which the amplitudes (energy) propagates
- while c is the propagation speed of the phase (in the direction $\boldsymbol{k}$)
- both are in general different and different from particle velocity

- consider the (linearized, $D/Dt \to \partial/\partial t$) layered model with $f \neq 0$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- take divergence of mom. equation, i.e. $\partial(1.eqn)/\partial x + \partial(2.eqn)/\partial y$

$$\frac{\partial}{\partial x}(1.\text{eqn}) : \frac{\partial}{\partial t}\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(fv) = -g\frac{\partial^2 h}{\partial x^2}$$
$$\frac{\partial}{\partial y}(2.\text{eqn}) : \frac{\partial}{\partial t}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}(fu) = -g\frac{\partial^2 h}{\partial y^2}$$

add both

$$\frac{\partial}{\partial t}\xi - \frac{\partial}{\partial x}(fv) + \frac{\partial}{\partial y}(fu) = -g\nabla^2 h$$
$$\frac{\partial}{\partial t}\xi - f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -g\nabla^2 h$$

with $\xi = \partial u/\partial x + \partial v/\partial y$ and for $f = const$

- consider the (linearized, $D/Dt \to \partial/\partial t$) layered model with $f \neq 0$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- take curl of mom. equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{\partial}{\partial x}(2.\text{eqn}) : \frac{\partial}{\partial x}\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(fu) = -g\frac{\partial^2 h}{\partial x \partial y}$$
$$\frac{\partial}{\partial y}(1.\text{eqn}) : \frac{\partial}{\partial y}\frac{\partial u}{\partial t} - \frac{\partial}{\partial y}(fv) = -g\frac{\partial^2 h}{\partial x \partial y}$$

subtract both

$$\frac{\partial}{\partial t}\zeta + \frac{\partial}{\partial x}(fu) + \frac{\partial}{\partial y}(fv) = 0$$
$$\frac{\partial}{\partial t}\zeta - f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

with $\zeta = \partial v/\partial x - \partial u/\partial y$ and for $f = const$

- thickness, curl and divergence for $f = const$

$$\begin{aligned} \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \\ \frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \\ \frac{\partial \xi}{\partial t} - f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= -g\boldsymbol{\nabla}^2 h \end{aligned}$$

with $\zeta = \partial v/\partial x - \partial u/\partial y$ and $\xi = \partial u/\partial x + \partial v/\partial y$

- time differentiate divergence and replace with curl and thickness eq.

$$\begin{aligned} \frac{\partial^2 \xi}{\partial t^2} - f\frac{\partial \zeta}{\partial t} &= -g\boldsymbol{\nabla}^2\frac{\partial h}{\partial t} \\ \frac{\partial^2}{\partial t^2}\xi + f^2\xi &= gH\boldsymbol{\nabla}^2\xi \\ \frac{\partial^2 \xi}{\partial t^2} + f^2\left(\xi - (gH/f^2)\boldsymbol{\nabla}^2\xi\right) &= 0 \\ \frac{\partial^2 \xi}{\partial t^2} + f^2\left(\xi - R^2\boldsymbol{\nabla}^2\xi\right) &= 0 \end{aligned}$$

with Rossby radius $R = \sqrt{gH}/|f|$

- combined thickness, curl and divergence eq. for $f = const$

$$\frac{\partial^2 \xi}{\partial t^2} + f^2\left(\xi - R^2\frac{\partial^2 \xi}{\partial x^2} - R^2\frac{\partial^2 \xi}{\partial y^2}\right) = 0$$

with Rossby radius $R = \sqrt{gH}/|f|$

- look for wave solutions

$$\xi(x, y, t) = \xi_0 \exp i(k_1 x + k_2 y - \omega t)$$

with complex constant ξ_0 which yields

$$\begin{aligned} (-i\omega)^2\xi_0\exp(\ldots) + f^2\left(1 - R^2(ik_1)^2 - R^2(ik_2)^2\right)\xi_0\exp(\ldots) &= 0 \\ -\omega^2 + f^2\left(1 + R^2k_1^2 + R^2k_2^2\right) &= 0 \end{aligned}$$

- this is a (plane wave) solution as long as ω satisfies

$$\omega = \pm\sqrt{f^2\left(1 + R^2k^2\right)}$$

with $k^2 = |\boldsymbol{k}|^2 = k_1^2 + k_2^2$

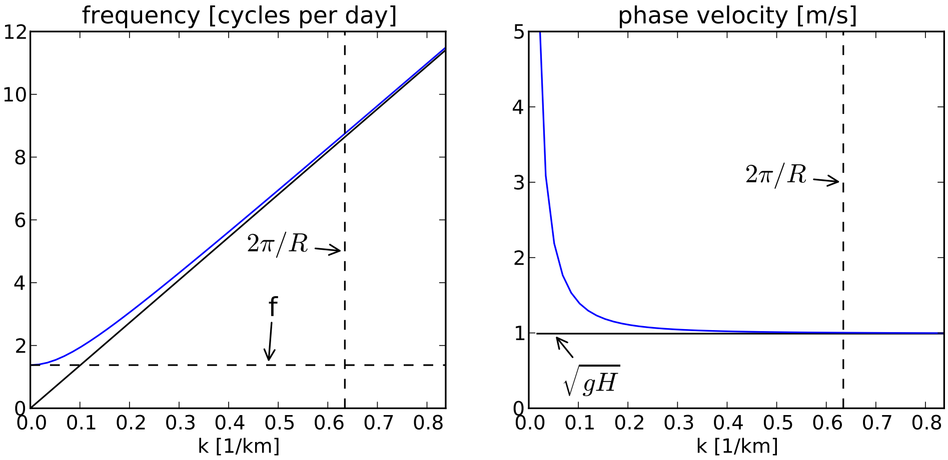
- gravity wave dispersion relation ($f \neq 0$ in blue, $f = 0$ in black)

$$\omega = \pm\sqrt{f^2\left(1 + R^2k^2\right)} \quad , \quad c = \pm\sqrt{f^2\left(1/k^2 + R^2\right)}$$

- different phase velocity $c = \omega/k$ for different $\boldsymbol{k}$ $\rightarrow$ dispersive wave
- short wave limit for $\lambda = 2\pi/k \ll R \rightarrow R^2k^2 \gg 1$

$$\omega \overset{Rk\rightarrow\infty}{=} \pm\sqrt{f^2R^2k^2} = \pm k\sqrt{gH} \quad , \quad c \overset{Rk\rightarrow\infty}{=} \pm\sqrt{gH}$$

$\rightarrow$ (non-dispersive) gravity waves without rotation (black lines)



- gravity wave dispersion relation ($f \neq 0$ in blue, $f = 0$ in black)

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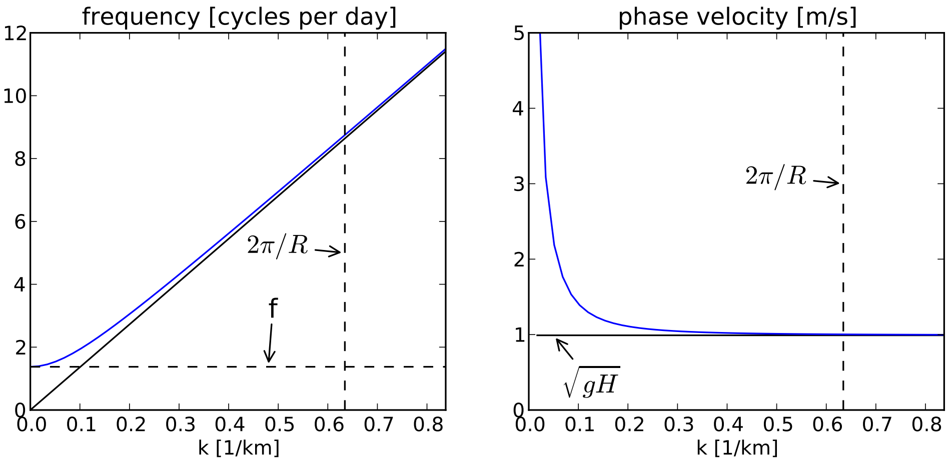
- different phase velocity $c = \omega/k$ for different $\boldsymbol{k}$ $\rightarrow$ dispersive wave
- long wave limit for $\lambda = 2\pi/k \gg R \rightarrow R^2k^2 \ll 1$

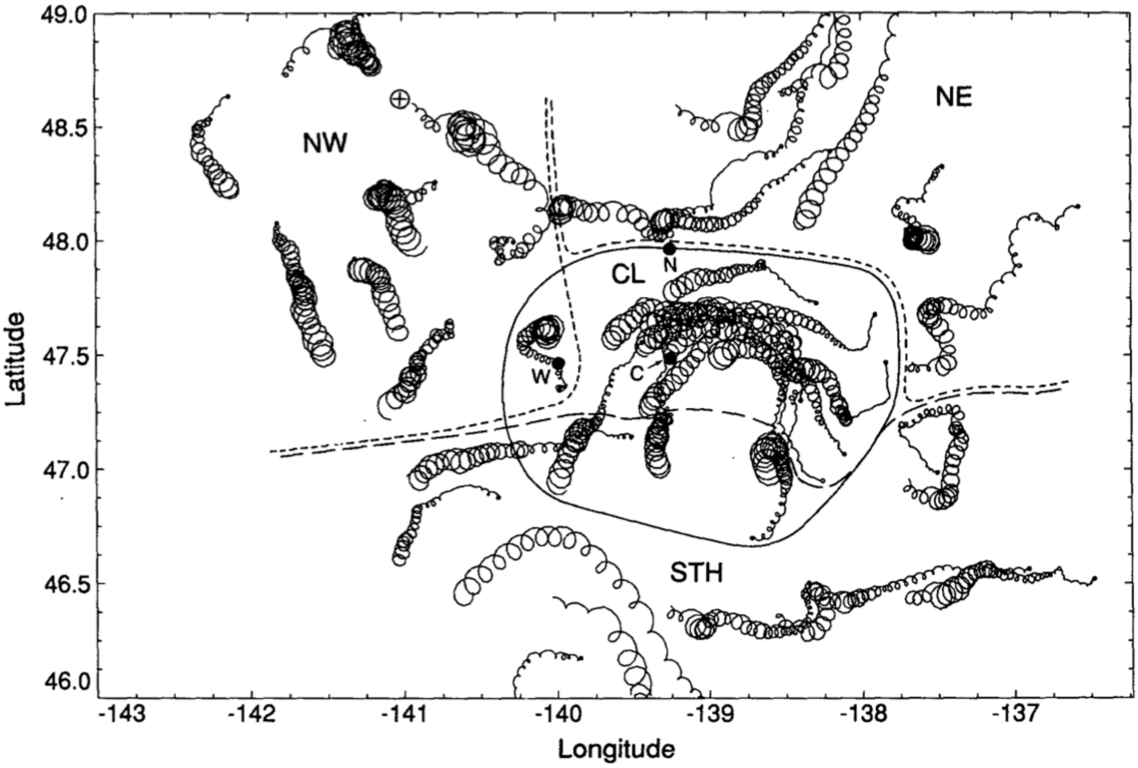
$$\omega \overset{Rk\rightarrow 0}{=} \pm f \quad , \quad c \overset{Rk\rightarrow 0}{=} \pm\infty$$

- these are inertial oscillations which also result from

$$\partial u/\partial t - fv = 0 \quad , \quad \partial v/\partial t + fu = 0$$



- trajectories of surface drifter $\rightarrow$ inertial oscillations



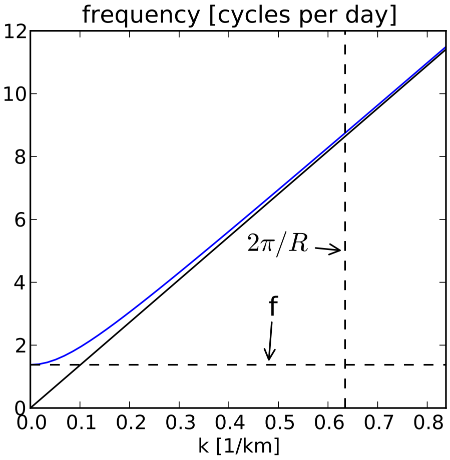
from d'Asaro et al 1995

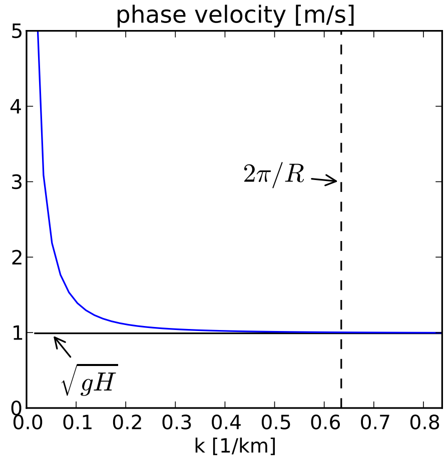
- gravity wave dispersion relation ($f \neq 0$ in blue, $f = 0$ in black)

$$\omega = \pm\sqrt{f^2\left(1 + R^2 k^2\right)}$$

- group velocity $\boldsymbol{c}_g = \partial\omega/\partial\boldsymbol{k}$ is given by

$$\boldsymbol{c}_g = \begin{pmatrix} \partial\omega/\partial k_1 \\ \partial\omega/\partial k_2 \end{pmatrix} = \pm \begin{pmatrix} \frac{1}{2}\left(f^2\left(1 + R^2k^2\right)\right)^{-1/2} f^2 R^2 2k_1 \\ \frac{1}{2}\left(f^2\left(1 + R^2k^2\right)\right)^{-1/2} f^2 R^2 2k_2 \end{pmatrix} = \frac{gH}{\omega}\boldsymbol{k}$$





- gravity wave dispersion relation ($f \neq 0$ in blue, $f = 0$ in black)

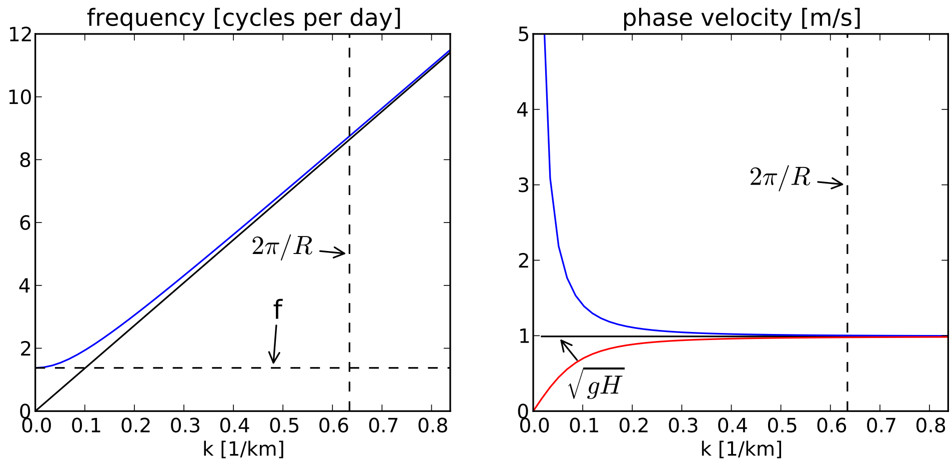
$$\omega = \pm\sqrt{f^2\left(1 + R^2 k^2\right)}$$

- group velocity is given by $\boldsymbol{c}_g = (gH/\omega)\boldsymbol{k}$ (red line for $f \neq 0$)
- short wave limit for $\lambda \ll R$

$$\omega \overset{\lambda \ll R}{=} \pm k\sqrt{gH} \quad \rightarrow \quad \boldsymbol{c}_g \overset{\lambda \ll R}{=} \pm\sqrt{gH}\,\boldsymbol{k}/k = c\,\boldsymbol{k}/k$$

- long wave limit for $\lambda \gg R$

$$\omega \overset{\lambda \gg R}{=} \pm f \quad \rightarrow \quad \boldsymbol{c}_g \overset{\lambda \gg R}{=} 0$$



- consider again the (linearized) layered model with $f \neq 0$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}\,,\quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}\,,\quad \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- suppose we have a solid boundary at $y = 0 \rightarrow v|_{y=0} = 0$
- look for solutions with $v = 0$ everywhere

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x}\,,\quad fu = -g\frac{\partial h}{\partial y}\,,\quad \frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

- combining the first and the last equation yields wave equation

$$\frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x^2} = 0$$

  with solution $h = A\exp i(kx - \omega t)$, but now $A = A(y)$
- gravity wave ($f = 0$) in x with phase velocity $c = \pm\sqrt{gH}$
- for y dependency of A we consider the second equation

- solid boundary at $y = 0$, look for solutions with $v = 0$ everywhere

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} \; , \; \frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0 \;\rightarrow\; \frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x^2} = 0$$

with solution $h = A(y) \exp i(kx - \omega t)$ and $\omega = \pm k\sqrt{gH}$

- for y dependency of A we consider the second equation
- assume wave $u = U(y) \exp i(kx - \omega t)$ with amplitude U from

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} \;\rightarrow\; -i\omega U \exp i(\ldots) = -gikA \exp i(\ldots) \;\rightarrow\; U = g\frac{kA}{\omega}$$

- using this in the second equation yields

$$fu = -g\frac{\partial h}{\partial y} \;\rightarrow\; (f/c)A = -A' \;\rightarrow\; A = A_0 e^{-f\,y/c} = A_0 e^{\pm y/R}$$

with $c = \omega/k = \pm\sqrt{gH}$ and with Rossby radius $R = \sqrt{gH}/|f|$

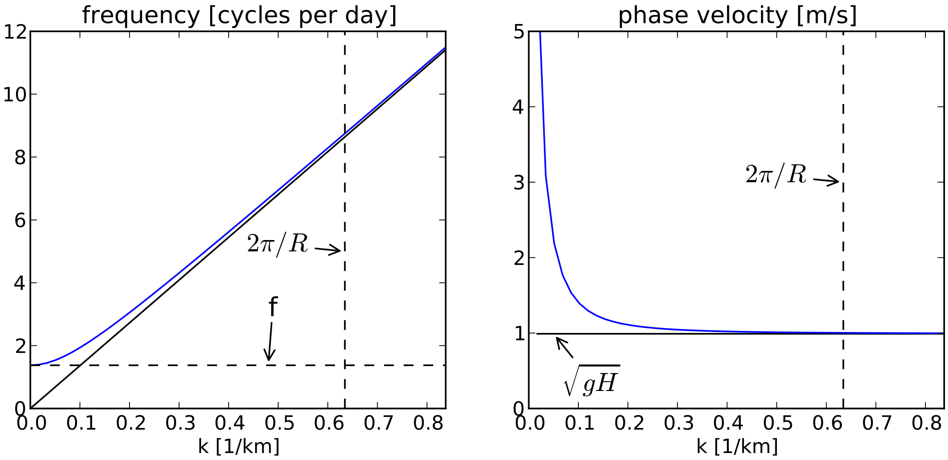
- only the decaying solution in y is reasonable
- Kelvin wave

- Kelvin wave along solid boundary at $y = 0$

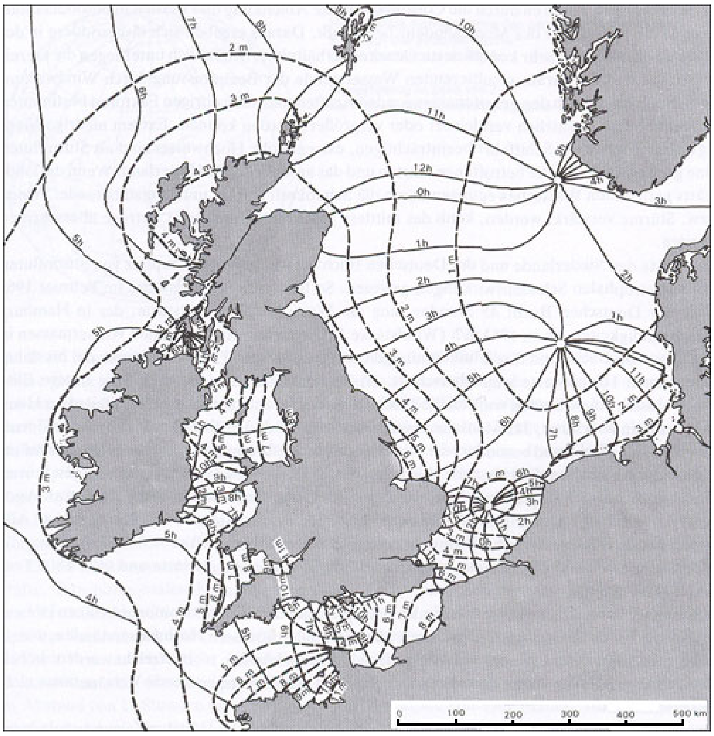
$$h = A_0 e^{\pm y/R} \exp i(kx - \omega t) \; , \; u = (gA_0/c)e^{\pm y/R} \exp i(kx - \omega t) \; , \; v = 0$$

and $\omega = \pm k\sqrt{gH}$ and with Rossby radius $R = \sqrt{gH}/|f|$

- only the decaying solution in y is reasonable
- works in the same way for boundary along x or any other direction



- tidal Kelvin wave in the North Sea



from Klett (2014)

- consider the (linearized) layered model with $f = const$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} \ , \quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y} \ , \quad \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- (linearized, $D/Dt \to \partial/\partial t$) potential vorticity equation

$$\frac{\partial q}{\partial t} = 0 \quad , \quad q = \frac{\zeta + f}{h} \approx \zeta - \frac{f}{H}h + f$$

- f in q for $f = const$ does not matter $\to q = \zeta - (f/H)h$
- consider as initial condition $\boldsymbol{u} = 0$ and h a step function such that

$$h|_{t=0} = \begin{cases} h_0, & \text{if } x < 0 \\ -h_0, & \text{if } x > 0 \end{cases} \quad \to \quad q_0 = q|_{t=0} = \begin{cases} -fh_0/H, & \text{if } x < 0 \\ fh_0/H, & \text{if } x > 0 \end{cases}$$

- using $q(t) = q_0$ steady state solution ($t \to \infty$) is given by

$$fv_\infty = g\frac{\partial h_\infty}{\partial x} \ , \quad fu_\infty = -g\frac{\partial h_\infty}{\partial y}$$

$$\to \quad q_\infty = \frac{g}{f}\frac{\partial^2 h_\infty}{\partial x^2} + \frac{g}{f}\frac{\partial^2 h_\infty}{\partial y^2} - \frac{f}{H}h_\infty = q_0$$

$$\to \boldsymbol{\nabla}^2 h_\infty - R^{-2}h_\infty = (f/g)q_0 \text{ with Rossby radius } R = \sqrt{gH}/|f|$$

- steady state solution ($t \to \infty$) is given by

$$\nabla^2 h_\infty - R^{-2} h_\infty = (f/g) q_0 = \begin{cases} -R^{-2} h_0, & \text{if } x < 0 \\ R^{-2} h_0, & \text{if } x > 0 \end{cases}$$

with Rossby radius $R = \sqrt{gH}/|f|$

- solution of h_∞ is given by

$$h(x)_\infty = \begin{cases} h_0(1 - e^{x/R}), & \text{if } x < 0 \\ -h_0(1 - e^{-x/R}), & \text{if } x > 0 \end{cases}$$

- since for $x < 0$ $h'_\infty = -h_0/R\, e^{x/R}$ and $h''_\infty = -h_0/R^2\, e^{x/R}$ and

$$h''_\infty - R^{-2} h_\infty = -h_0 R^{-2} e^{x/R} - R^{-2} h_0 (1 - e^{x/R}) = -R^{-2} h_0$$

- since for $x > 0$ $h'_\infty = -h_0/R\, e^{-x/R}$ and $h''_\infty = h_0/R^2\, e^{-x/R}$ and

$$h''_\infty - R^{-2} h_\infty = h_0 R^{-2} e^{-x/R} + R^{-2} h_0 (1 - e^{-x/R}) = R^{-2} h_0$$

- initial and steady state solution of h are given by

$$h|_{t=0} = \begin{cases} h_0, & \text{if } x < 0 \\ -h_0, & \text{if } x > 0 \end{cases} \quad , \quad h|_\infty = \begin{cases} h_0(1 - e^{x/R}), & \text{if } x < 0 \\ -h_0(1 - e^{-x/R}), & \text{if } x > 0 \end{cases}$$

with Rossby radius $R = \sqrt{gH}/|f|$

- velocities from $fv_\infty = g\partial h_\infty/\partial x$ and $fu_\infty = -g\partial h_\infty/\partial y$

$$u_\infty = 0 \quad , \quad v_\infty = (g/f) \begin{cases} -h_0/Re^{x/R}, & \text{if } x < 0 \\ -h_0/Re^{-x/R}, & \text{if } x > 0 \end{cases} = -\frac{gh_0}{fR} e^{-|x|/R}$$