Dynamische und regionale Ozeanographie
WS 2014/15

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June 30, 2015

Lecture # 9

Recapitulation

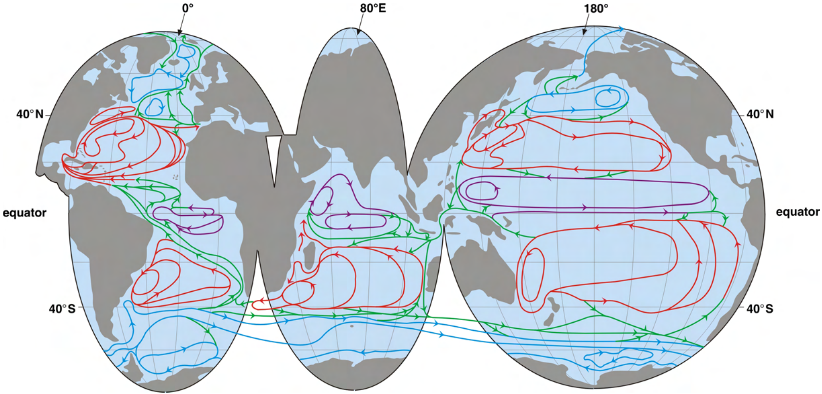
- Elementarstromsystem
- Ekman transport
- Ekman pumping
- Sverdrup transport
- Sverdrup meets Ekman

Wind driven circulation

- Western boundary currents

oral examination Tuesday July 7., 2015

- 13:00 Tabea Kilchling
- 13:30 Isabell Hochfeld
- 14:00 Lucas Schmidt
- 14:30 Annika Buck
- 15:00 Elena Hirschhoff
- 15:30 Heninng Dorff
- 16:00 Jerome Sauer
- 16:30 Carolin Meier ?
- 17:00 Sophie Specht ?
- 17:30 Anna Wünsche ?



- Schematic of the near-surface circulation (after Schmitz 1996). Subtropical gyres are red, subpolar and polar gyres blue equatorial gyres magenta, Antarctic Circumpolar Current is blue green lines represent exchange between basins and gyres

- momentum equation in vector form for $Ro \ll 1$

$$f\boldsymbol{k} \times \boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla}_h p + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \quad \text{with} \quad \boldsymbol{k} \times \boldsymbol{u} = (-v, u, \not{0})$$

- $\boldsymbol{\tau} = (\tau^x, \tau^y)$ is a stress vector with $\boldsymbol{\tau}(z=0) = \boldsymbol{\tau}^a$ where $\boldsymbol{\tau}^a$ is the surface wind stress in $\mathrm{N/m^2}$ acting on the ocean

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- $\boldsymbol{\tau} = (\tau^x, \tau^y)$ is a stress vector with $\boldsymbol{\tau}(z=0) = \boldsymbol{\tau}^a$ where $\boldsymbol{\tau}^a$ is the surface wind stress in N/m^2 acting on the ocean
- split the flow into geostrophic and frictional (Ekman) components, $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$), governed by

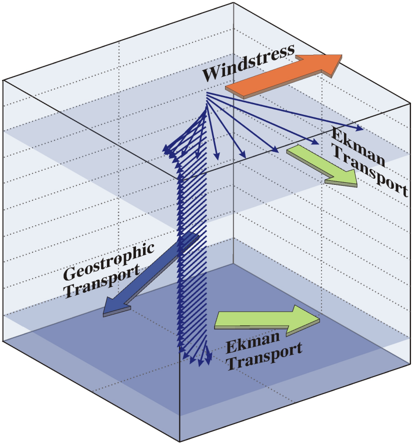
$$f\boldsymbol{k} \times \boldsymbol{u}_G = -\frac{1}{\rho_0}\boldsymbol{\nabla}_h p \qquad \text{and} \qquad f\boldsymbol{k} \times \boldsymbol{u}_E = \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}$$

and the same for continuity equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_G + \frac{\partial w_G}{\partial z} = 0 \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u}_E + \frac{\partial w_E}{\partial z} = 0$$

- sum $\boldsymbol{u}_G + \boldsymbol{u}_E$ satisfies full momentum and continuity equation

- Elementarstromsystem (for $\rho = const$)
- $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$)
  surface and bottom Ekman layers superimposed on geostrophic flow



- vertically integrated velocity

$$\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u}\, dz = \int_{-h}^{0} (\boldsymbol{u}_G + \boldsymbol{u}_E)\, dz = \boldsymbol{U}_G + \boldsymbol{U}_E$$

- vertically integrated velocity

$$\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u} \; dz = \int_{-h}^{0} (\boldsymbol{u}_G + \boldsymbol{u}_E) \; dz = \boldsymbol{U}_G + \boldsymbol{U}_E$$

- with the (total) transport vector $\boldsymbol{U}$, dimension $\mathrm{m^2 s^{-1}}$

- vertically integrated velocity

$$\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u} \, dz = \int_{-h}^{0} (\boldsymbol{u}_G + \boldsymbol{u}_E) \, dz = \boldsymbol{U}_G + \boldsymbol{U}_E$$

- with the (total) transport vector $\boldsymbol{U}$, dimension $\mathrm{m}^2\mathrm{s}^{-1}$
- transport by the geostrophic velocity $\rightarrow$ geostrophic transport $\boldsymbol{U}_G$
  transport by the Ekman velocity $\rightarrow$ Ekman transport $\boldsymbol{U}_E$

$$f\boldsymbol{k} \times \boldsymbol{u}_E = \frac{1}{\rho_0} \frac{\partial \boldsymbol{\tau}}{\partial z}$$

- vertically integrated velocity

$$\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u}\, dz = \int_{-h}^{0} (\boldsymbol{u}_G + \boldsymbol{u}_E)\, dz = \boldsymbol{U}_G + \boldsymbol{U}_E$$

- with the (total) transport vector $\boldsymbol{U}$, dimension $\mathrm{m}^2\mathrm{s}^{-1}$
- transport by the geostrophic velocity → geostrophic transport $\boldsymbol{U}_G$
  transport by the Ekman velocity → Ekman transport $\boldsymbol{U}_E$

$$f\boldsymbol{k} \times \boldsymbol{u}_E = \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}$$

$$f\boldsymbol{k} \times \int_{-h}^{0} \boldsymbol{u}_E\, dz = \frac{1}{\rho_0}\left(\boldsymbol{\tau}(z=0) - \boldsymbol{\tau}(z=-h)\right)$$

- vertically integrated velocity

$$\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u}\, dz = \int_{-h}^{0} (\boldsymbol{u}_G + \boldsymbol{u}_E)\, dz = \boldsymbol{U}_G + \boldsymbol{U}_E$$

- with the (total) transport vector $\boldsymbol{U}$, dimension $\mathrm{m^2 s^{-1}}$
- transport by the geostrophic velocity $\rightarrow$ geostrophic transport $\boldsymbol{U}_G$
  transport by the Ekman velocity $\rightarrow$ Ekman transport $\boldsymbol{U}_E$

$$\begin{aligned} f\boldsymbol{k} \times \boldsymbol{u}_E &= \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \\ f\boldsymbol{k} \times \int_{-h}^{0} \boldsymbol{u}_E\, dz &= \frac{1}{\rho_0}\left(\boldsymbol{\tau}(z=0) - \boldsymbol{\tau}(z=-h)\right) \\ f\boldsymbol{k} \times \boldsymbol{U}_E &= \frac{1}{\rho_0}\left(\boldsymbol{\tau}^a - \boldsymbol{\tau}_b\right) \\ \boldsymbol{U}_E &= -\frac{1}{f\rho_0}\boldsymbol{k} \times \left(\boldsymbol{\tau}^a - \boldsymbol{\tau}_b\right) \end{aligned}$$

with surface wind stress $\boldsymbol{\tau}^a$ and bottom stress $\boldsymbol{\tau}_b$

- vertically integrated velocity

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- with the (total) transport vector $\boldsymbol{U}$, dimension $\mathrm{m}^2\mathrm{s}^{-1}$
- transport by the geostrophic velocity → geostrophic transport $\boldsymbol{U}_G$
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  with surface wind stress $\boldsymbol{\tau}^a$ and bottom stress $\boldsymbol{\tau}_b$
- split $\boldsymbol{U}_E$ into surface and bottom Ekman transport

- vertically integrated velocity $\boldsymbol{U} = \boldsymbol{U}_G + \boldsymbol{U}_E$ with geostrophic transport $\boldsymbol{U}_G$ and Ekman transport $\boldsymbol{U}_E$ given by

$$\boldsymbol{U}_E = -\frac{1}{f\rho_0}\boldsymbol{k} \times (\boldsymbol{\tau}^a - \boldsymbol{\tau}_b)$$

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$$\boldsymbol{U}_E = -\frac{1}{f\rho_0}\boldsymbol{k} \times (\boldsymbol{\tau}^a - \boldsymbol{\tau}_b) \equiv \boldsymbol{U}_E^{top} + \boldsymbol{U}_E^{bot}$$

  with surface wind stress $\boldsymbol{\tau}^a$ and bottom stress $\boldsymbol{\tau}_b$

- split into surface Ekman transport in surface Ekman layer

$$\boldsymbol{U}_E^{top} = -\frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}^a$$

  orthogonal to wind stress direction (to the right for $f > 0$)

  does not depend on parameterisation of $\boldsymbol{\tau}$ in the interior

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- and bottom Ekman transport in bottom Ekman layer

$$\boldsymbol{U}_E^{bot} = \frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}_b$$

  depends on parameterisation of $\boldsymbol{\tau}$ in the interior

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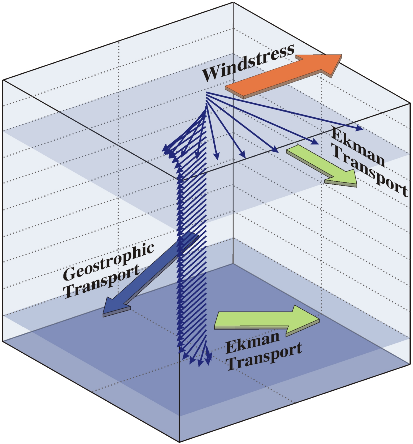
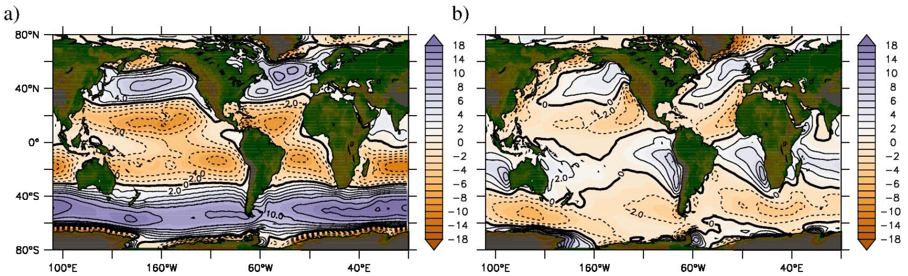
  surface and bottom Ekman layers superimposed on geostrophic flow



- zonal (left) and meridional component (right) of $\boldsymbol{\tau}^a$ in $10^{-2}\,\mathrm{N/m^2}$



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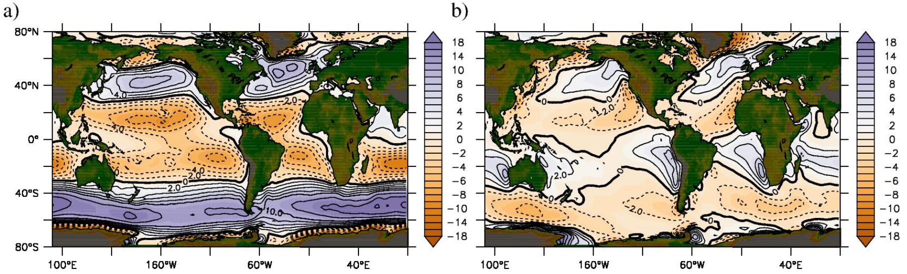


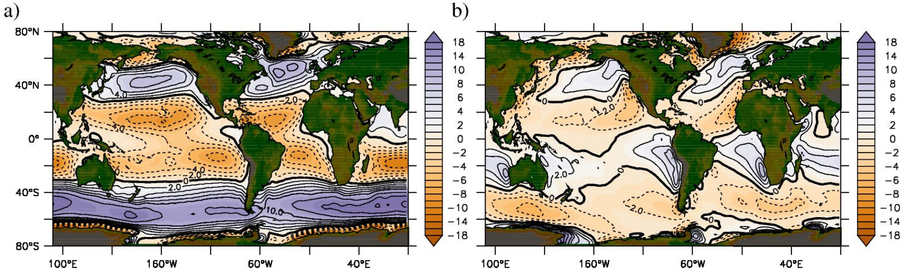
- surface Ekman transport in surface Ekman layer

$$\boldsymbol{U}_E^{top} = -\frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}^a$$

orthogonal to wind stress direction (to the right for $f > 0$)

- equatorward in west wind region poleward in trade wind region

- zonal (left) and meridional component (right) of $\boldsymbol{\tau}^a$ in $10^{-2}\,\mathrm{N/m^2}$



- surface Ekman transport in surface Ekman layer

$$\boldsymbol{U}_E^{top} = -\frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}^a$$

orthogonal to wind stress direction (to the right for $f > 0$)

- equatorward in west wind region poleward in trade wind region
- convergence between west wind and trade wind region
- divergence at high latitude and at equator

- integrating the continuity equation for $\boldsymbol{u}_E$ and w_E from z to $z = 0$

$$\nabla \cdot \boldsymbol{u}_E + \frac{\partial w_E}{\partial z} = 0$$

yields the vertical Ekman velocity

$$\int_z^0 \nabla \cdot \boldsymbol{u}_E \, dz + \cancel{w_E(z=0)} - w_E(z) = 0 \;\rightarrow\; w_E(z) = \nabla \cdot \int_z^0 \boldsymbol{u}_E \, dz$$

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- since $\boldsymbol{u}_E \approx 0$ below Ekman depth $D \approx 50\,\mathrm{m}$

$$w_E|_{z<-D} \approx \boldsymbol{\nabla} \cdot \int_{z<-D}^0 \boldsymbol{u}_E \, dz = \boldsymbol{\nabla} \cdot \boldsymbol{U}_E^{top} = -\boldsymbol{\nabla} \cdot \frac{1}{f \rho_0} \boldsymbol{k} \times \boldsymbol{\tau}^a$$

with Ekman pumping $w_E|_{z<-D}$

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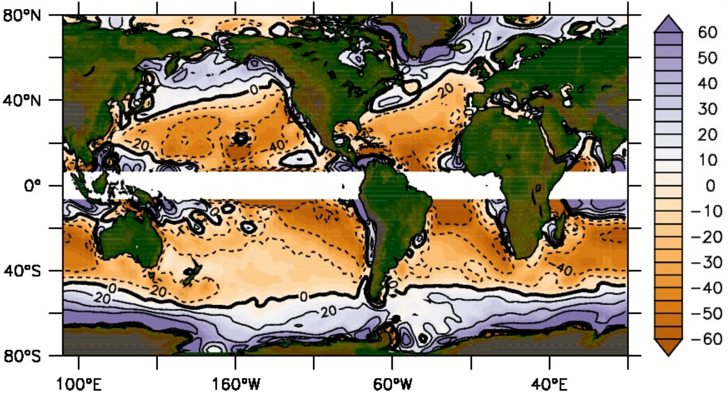
with Ekman pumping $w_E\big|_{z<-D}$

- Ekman pumping w_E in m per year

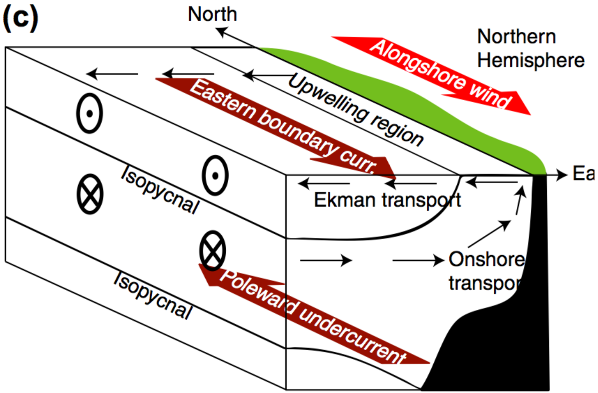
$$w_E\big|_{z<-D} \approx \boldsymbol{\nabla} \cdot \boldsymbol{U}_E^{top} = \boldsymbol{k} \times \boldsymbol{\nabla} \cdot \frac{\boldsymbol{\tau}^a}{\rho_0 f}$$

with Ekman depth $D \approx 50\,\mathrm{m}$ (depends on A_v)



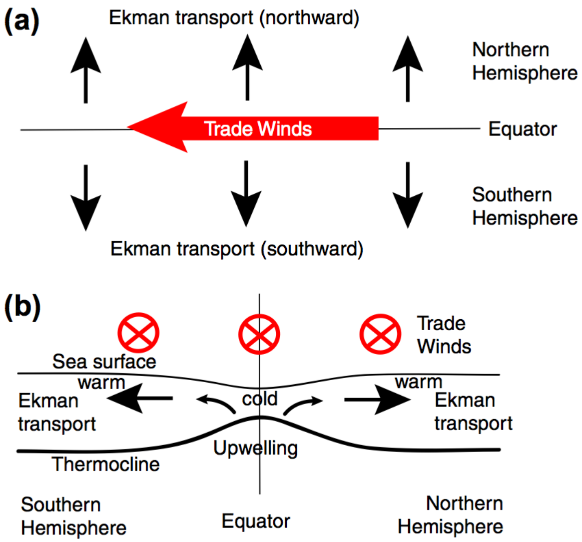
- Ekman transport $\boldsymbol{U}_E^{top}$ and pumping w_E do not depend on A_v

- coastal upwelling



from Talley et al 2011

- equatorial upwelling



from Talley et al 2011

- momentum equation in planetary geostrophic approximation

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- neglect sea surface height $z = \zeta \rightarrow$ assume rigid lid at $z = 0$
- assume flat bottom at $z = -h = const$
- vertically integrate from bottom to surface

$$\rho_0 f\boldsymbol{k}\times\boldsymbol{U} = -\boldsymbol{\nabla}_h \int_{-h}^{0} p dz + \boldsymbol{\tau}_a - \boldsymbol{\tau}_b$$

with transport $\boldsymbol{U} = \int_{-h}^{0}\boldsymbol{u}dz$, surface and bottom stress $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$

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$$\rho_0 f\boldsymbol{k} \times \boldsymbol{U} = -\boldsymbol{\nabla}_h \int_{-h}^{0} p dz + \boldsymbol{\tau}_a - \boldsymbol{\tau}_b$$

  with transport $\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u} dz$, surface and bottom stress $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$
- take curl and it follows the famous Sverdrup relation

$$\rho_0 \beta V \quad = \quad \boldsymbol{k} \times \boldsymbol{\nabla} \cdot (\boldsymbol{\tau}_a - \boldsymbol{\tau}_b)$$

  where $\boldsymbol{\tau}_b$ is often neglected
- depth integrated transport V calculated from wind stress curl only

- since $\boldsymbol{\nabla}_h \cdot \boldsymbol{U} = 0$ introduce volume transport streamfunction, with

$$U = -\frac{\partial \psi}{\partial y} \quad , \quad V = \frac{\partial \psi}{\partial x} \quad \rightarrow \quad \boldsymbol{U} = \boldsymbol{k} \times \boldsymbol{\nabla} \psi$$

transport $\boldsymbol{U}$ is parallel to contour lines of ψ

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- ψ determines transport perpendicular to or "across" section $A \rightarrow B$
- Sverdrup relation becomes

$$\rho_0 \beta \, V = \rho_0 \beta \frac{\partial \psi}{\partial x} = \boldsymbol{k} \times \nabla \cdot \boldsymbol{\tau}_a$$

for $\boldsymbol{\tau}_b = 0$

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- Sverdrup relation becomes

$$\rho_0 \beta V = \rho_0 \beta \frac{\partial \psi}{\partial x} = \boldsymbol{k} \times \nabla \cdot \boldsymbol{\tau}_a$$

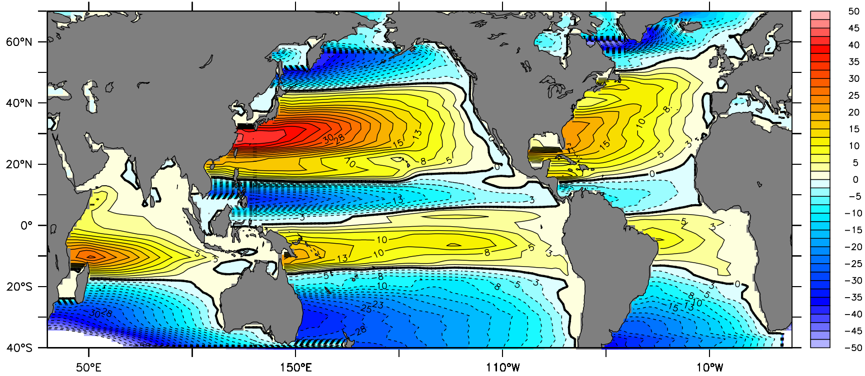
  for $\boldsymbol{\tau}_b = 0$

- integration from eastern boundary $(x = x_E)$ where $\psi(x_E) = 0$

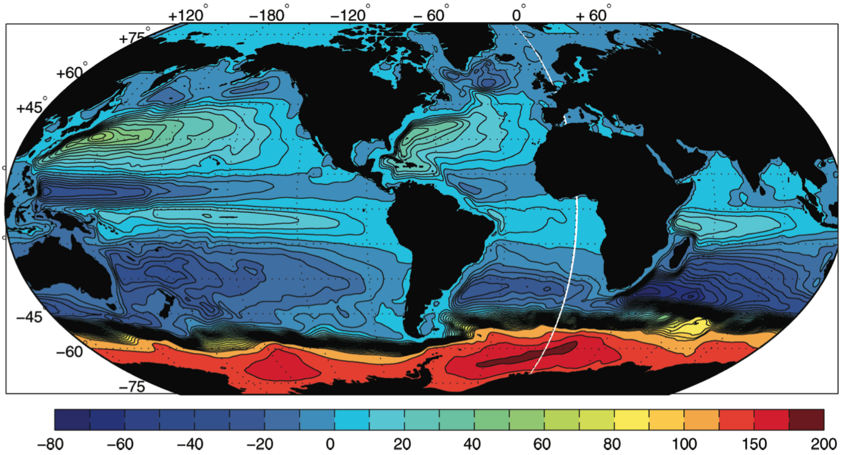
$$\psi(x, y) \quad = \quad -\frac{1}{\rho_0 \beta} \int_x^{x_e} \boldsymbol{k} \times \nabla \cdot \boldsymbol{\tau}_a \, dx$$

- $\psi(x_E) = 0$ along east eastern boundary but not at western boundary
  $\rightarrow$ western boundary current not included

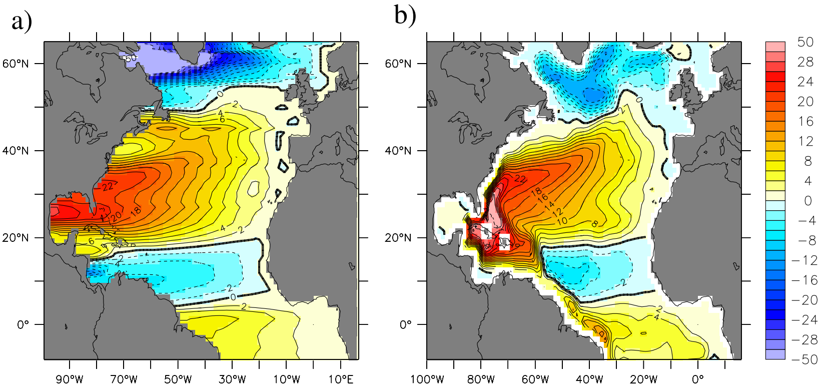
- $\psi = -1/(\rho_0\beta)\int_x^{x_e} \boldsymbol{k}\times\boldsymbol{\nabla}\cdot\boldsymbol{\tau}^a\,dx$ from realistic wind stress in $10^6\,\mathrm{m}^3/\mathrm{s} \equiv 1\,\mathrm{Sv}$



- ψ in a global state estimate in $10^6\,\mathrm{m}^3/\mathrm{s} \equiv 1\,\mathrm{Sv}$



- Streamfunction ψ in *Sv* $= 10^6\,\mathrm{m}^3/\mathrm{s}$ from simple Sverdrup relation
- Streamfunction ψ for a realistic model of the Atlantic Ocean



- simple Sverdrup relation works surprisingly well

Recapitulation

Wind driven circulation

- vertically integrated momentum equation

$$
\begin{aligned}
-\rho_0 f V &= -\frac{\partial}{\partial x}\int_{-h}^{0} p dz + \tau_a^x \equiv -\frac{\partial P}{\partial x} + \tau_a^x \\
\rho_0 f U &= -\frac{\partial}{\partial y}\int_{-h}^{0} p dz + \tau_a^y \equiv -\frac{\partial P}{\partial y} + \tau_a^y
\end{aligned}
$$

- vertically integrated momentum equation

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- split in Ekman transport $\boldsymbol{U}_E$ und geostrophic transport $\boldsymbol{U}_G$

$$-\rho_0 fV \equiv -\rho_0 f\,(V_G + V_E) = -\frac{\partial P}{\partial x} + \tau_a^x$$
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$$

- split in Ekman transport $\boldsymbol{U}_E$ und geostrophic transport $\boldsymbol{U}_G$

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-\rho_0 fV \equiv -\rho_0 f\,(V_G + V_E) &= -\frac{\partial P}{\partial x} + \tau_a^x \\
\rho_0 fU \equiv \rho_0 f\,(U_G + U_E) &= -\frac{\partial P}{\partial y} + \tau_a^y
\end{aligned}
$$

- with Ekman transport

$$
-\rho_0 fV_E = \tau_a^x\ ,\ \rho_0 fU_E = \tau_a^y \ \rightarrow\ \rho_0 f\,\boldsymbol{k}\times\boldsymbol{U}_E = \boldsymbol{\tau}_a \ \rightarrow\ \rho_0 f\,\boldsymbol{U}_E = -\boldsymbol{k}\times\boldsymbol{\tau}_a
$$

- vertically integrated momentum equation

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-\rho_0 fV \equiv -\rho_0 f\left(V_G + V_E\right) &= -\frac{\partial P}{\partial x} + \tau_a^x \\
\rho_0 fU \equiv \rho_0 f\left(U_G + U_E\right) &= -\frac{\partial P}{\partial y} + \tau_a^y
\end{aligned}
$$

- with Ekman transport

$$
-\rho_0 fV_E = \tau_a^x \ , \ \rho_0 fU_E = \tau_a^y \ \rightarrow \ \rho_0 f \boldsymbol{k} \times \boldsymbol{U}_E = \boldsymbol{\tau}_a \ \rightarrow \ \rho_0 f \boldsymbol{U}_E = -\boldsymbol{k} \times \boldsymbol{\tau}_a
$$

- and with geostrophic transport

$$
-\rho_0 fV_G = -\frac{\partial P}{\partial x} \ , \ \rho_0 fU_G = -\frac{\partial P}{\partial y} \ \rightarrow \ \rho_0 f \boldsymbol{U}_G = \boldsymbol{k} \times \boldsymbol{\nabla}_h P
$$

- vertically integrated momentum equation

$$
\begin{aligned}
-\rho_0 f V &= -\frac{\partial}{\partial x}\int_{-h}^{0} p dz + \tau_a^x \equiv -\frac{\partial P}{\partial x} + \tau_a^x \\
\rho_0 f U &= -\frac{\partial}{\partial y}\int_{-h}^{0} p dz + \tau_a^y \equiv -\frac{\partial P}{\partial y} + \tau_a^y
\end{aligned}
$$

- split in Ekman transport $\boldsymbol{U}_E$ und geostrophic transport $\boldsymbol{U}_G$

$$
\begin{aligned}
-\rho_0 f V \equiv -\rho_0 f\left(V_G + V_E\right) &= -\frac{\partial P}{\partial x} + \tau_a^x \\
\rho_0 f U \equiv \rho_0 f\left(U_G + U_E\right) &= -\frac{\partial P}{\partial y} + \tau_a^y
\end{aligned}
$$

- with Ekman transport

$$
-\rho_0 f V_E = \tau_a^x\ ,\ \rho_0 f U_E = \tau_a^y \ \rightarrow\ \rho_0 f\, \boldsymbol{k} \times \boldsymbol{U}_E = \boldsymbol{\tau}_a \ \rightarrow\ \rho_0 f\, \boldsymbol{U}_E = -\boldsymbol{k} \times \boldsymbol{\tau}_a
$$

- and with geostrophic transport

$$
-\rho_0 f V_G = -\frac{\partial P}{\partial x}\ ,\ \rho_0 f U_G = -\frac{\partial P}{\partial y} \ \rightarrow\ \rho_0 f\, \boldsymbol{U}_G = \boldsymbol{k} \times \boldsymbol{\nabla}_h P
$$

- Ekman transport + geostr. transport = Sverdrup transport

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- both transports are divergent

$$\boldsymbol{\nabla}_h \cdot \boldsymbol{U}_E \quad = \quad -\boldsymbol{\nabla}_h \cdot \boldsymbol{k} \times \frac{\boldsymbol{\tau}_a}{\rho_0 f} = \boldsymbol{k} \times \boldsymbol{\nabla}_h \cdot \frac{\boldsymbol{\tau}_a}{\rho_0 f} = w_E$$

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$$-\rho_0 f V_G = -\frac{\partial P}{\partial x} \quad , \quad \rho_0 f U_G = -\frac{\partial P}{\partial y} \quad , \quad \rho_0 f \boldsymbol{U}_G = \boldsymbol{k} \times \boldsymbol{\nabla}_h P$$

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\end{aligned}$$

- Ekman transport + geostr. transport = Sverdrup transport
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\end{aligned}$$

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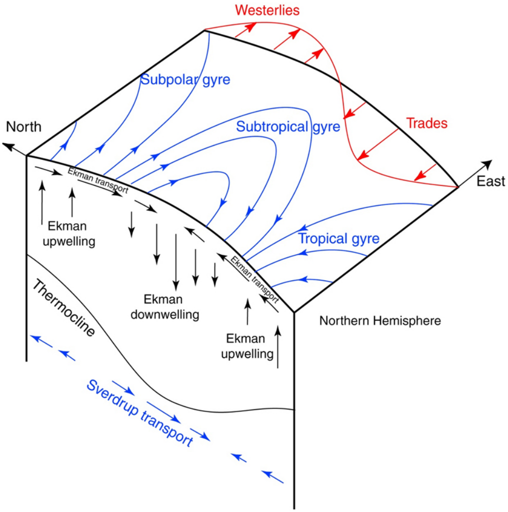
- but the total transport $\boldsymbol{U} = \boldsymbol{U}_E + \boldsymbol{U}_G$ is non-divergent

$$\boldsymbol{\nabla}_h \cdot \boldsymbol{U} = 0 \quad \rightarrow \quad w_E^{top} = \frac{\beta}{f} V_G$$

Ekman pumping generates southward geostr. transport (for $f > 0$)

- Ekman pumping generates southward geostr. transport (for $f > 0$)



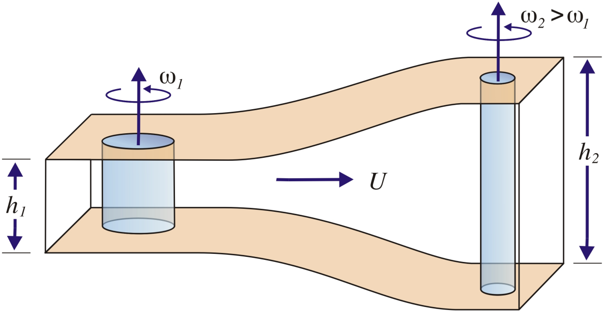
from Talley et al 2011

- Sverdrup relation follows from potential vorticity conservation
- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q \approx \zeta - \frac{f_0}{H} h + f$$

q is conserved for fluid parcels in single layer

- w_E lead to vortex stretching and meridional motion



- momentum equation in planetary geostrophic approximation

$$f\,\boldsymbol{k} \times \boldsymbol{u} = -\boldsymbol{\nabla}_h p + \frac{\partial \boldsymbol{\tau}}{\partial z} + A_h \boldsymbol{\nabla}_h^2 \boldsymbol{u} - R\,\boldsymbol{u}$$

with stress vector $\boldsymbol{\tau}$ connecting to surface wind stress
with lateral friction related to the lateral turbulent viscosity A_h
and with turbulent Rayleigh (bottom) friction related to R

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- vertically integrating from flat bottom to surface (assume $\boldsymbol{\tau}_b = 0$)

$$f\boldsymbol{k} \times \int_{-h}^{0} \boldsymbol{u} dz \;=\; -\int_{-h}^{0} \boldsymbol{\nabla}_h p dz + \boldsymbol{\tau}_a + \int_{-h}^{0} (A_h \boldsymbol{\nabla}_h^2 - R)\boldsymbol{u} dz$$

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- for $h = const$ vertical integration and $\boldsymbol{\nabla}$ commute

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  with transport $\boldsymbol{U} = \int_{-h}^{0}\boldsymbol{u}dz$
- use transport streamfunction ψ with $\boldsymbol{U} = \boldsymbol{k}\times\boldsymbol{\nabla}\psi$

$$-f\boldsymbol{\nabla}_h\psi = -\boldsymbol{\nabla}_h\int_{-h}^{0}pdz + \boldsymbol{\tau}_a + (A_h\boldsymbol{\nabla}_h^2 - R)\boldsymbol{k}\times\boldsymbol{\nabla}\psi$$

- vertically integrated momentum equation

$$-f\nabla_h\psi = -\nabla_h\int_{-h}^{0} pdz + \boldsymbol{\tau}_a + (A_h\nabla_h^2 - R)\boldsymbol{k}\times\nabla\psi$$

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with transport $\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u} dz$ and streamfunction $\boldsymbol{U} = \boldsymbol{k}\times\nabla\psi$

- now take curl $(\boldsymbol{k}\times\nabla)\cdot$ of momentum equation

$$-(\boldsymbol{k}\times\nabla)\cdot f\nabla_h\psi = \boldsymbol{k}\times\nabla\cdot\boldsymbol{\tau}_a + (A_h\nabla_h^2 - R)(\boldsymbol{k}\times\nabla)\cdot(\boldsymbol{k}\times\nabla)\psi$$

since $(\boldsymbol{k}\times\nabla)\cdot\nabla_h\int_{-h}^{0} p dz = 0$

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- with

$$-(\boldsymbol{k}\times\boldsymbol{\nabla})\cdot f\boldsymbol{\nabla}_h\psi = -f(\boldsymbol{k}\times\boldsymbol{\nabla})\cdot\boldsymbol{\nabla}_h\psi - \boldsymbol{\nabla}_h\psi\cdot(\boldsymbol{k}\times\boldsymbol{\nabla})f =$$

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and with $(\boldsymbol{k}\times\boldsymbol{\nabla})\cdot(\boldsymbol{k}\times\boldsymbol{\nabla})\psi = \boldsymbol{\nabla}_h^2\psi$

- the Stommel/Munk equation for flat bottom follows as

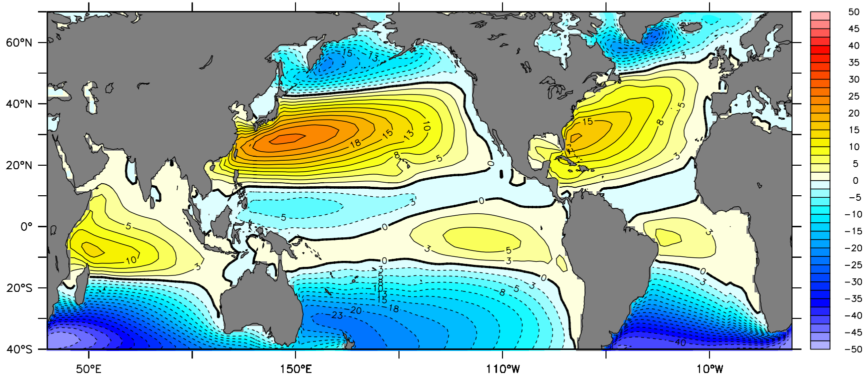
$$\beta\frac{\partial\psi}{\partial x} = \boldsymbol{k}\times\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_a + (A_h\boldsymbol{\nabla}_h^2 - R)\boldsymbol{\nabla}_h^2\psi$$

first part identical to Sverdrup relation, friction related to A_h and R closes circulation at western boundary

- numerical solution of Stommel's equation

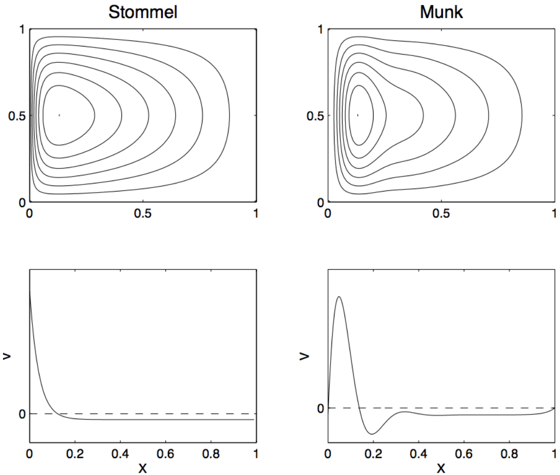
$$\beta \frac{\partial \psi}{\partial x} = \boldsymbol{k} \times \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_a - R \boldsymbol{\nabla}_h^2 \psi$$

with realistic wind stress $\boldsymbol{\tau}_a$ (and $A_h = 0$)



- Henry Melson Stommel, * 1920 in Wilmington (USA) †1992 in Boston (USA), oceanographer.

- consider a wind stress of the form $\boldsymbol{\tau}_a = (-\tau_0 \cos \frac{\pi y}{B}, 0)$ with $A_h = 0 \rightarrow$ Stommel's equation (left) and $R = 0 \rightarrow$ Munk's equation (right)



from Vallis 2006

- boundary layer scaling

$$\beta \frac{\partial \psi}{\partial x} = \boldsymbol{k} \times \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_a + (A_h \boldsymbol{\nabla}_h^2 - R) \boldsymbol{\nabla}_h^2 \psi$$

- balance bottom friction $R\boldsymbol{\nabla}_h^2\psi$ and planetary vorticity $\beta\partial\psi/\partial x$

$$\begin{aligned} \beta\psi/L &\sim R\psi/L^2 \\ L &\sim R/\beta \end{aligned}$$

Stommel's boundary layer with $R/\beta \approx 50 - 100\,\mathrm{km}$

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- balance lateral friction $A_h\boldsymbol{\nabla}_h^4\psi$ and planetary vorticity $\beta\partial\psi/\partial x$

$$\begin{aligned} \beta\psi/L &\sim A_h\psi/L^4 \\ L &\sim (A_h/\beta)^{1/3} \end{aligned}$$

  Munk's boundary layer with $(A_h/\beta)^{1/3}$

- $(A_h/\beta)^{1/3}$ is often used to choose value for A_h in numerical models

- why is the western boundary current in the west?
- dominant balance in the western boundary current regime

$$\beta \frac{\partial \psi}{\partial x} = \beta V \approx \cancel{\boldsymbol{k} \times \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_a} - R \boldsymbol{\nabla}_h^2 \psi \approx -R \frac{\partial^2 \psi}{\partial x^2} = -R \frac{\partial V}{\partial x}$$

between bottom friction and change in planetary vorticity

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  between bottom friction and change in planetary vorticity
- since $V < 0$ in the interior of the subtropical gyre
  $V > 0$ in the western boundary $\rightarrow \beta V > 0 \rightarrow R \partial V / \partial x < 0$

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- for eastern boundary layer V increases to the east
  $\rightarrow$ no eastern boundary layer
- westward Rossby waves propagate to the west
- they are reflected at the western boundary as short Rossby waves with eastward group velocity

- why is the western boundary current in the west?
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$$\beta\frac{\partial\psi}{\partial x} = \beta V \approx \cancel{\boldsymbol{k}\times\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_a} - R\boldsymbol{\nabla}_h^2\psi \approx -R\frac{\partial^2\psi}{\partial x^2} = -R\frac{\partial V}{\partial x}$$

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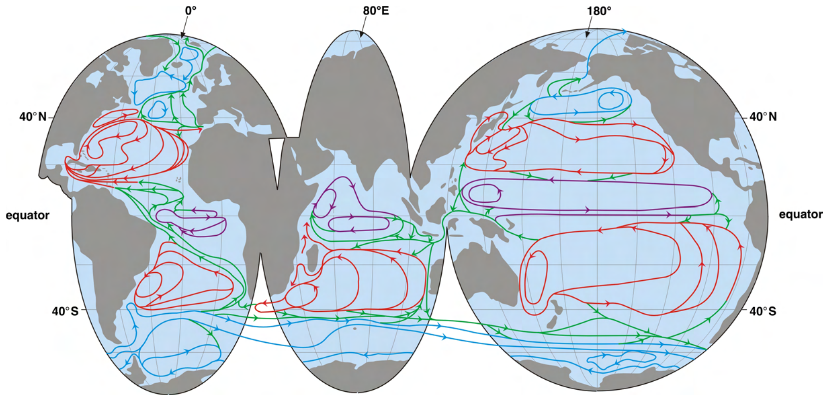
- since $V < 0$ in the interior of the subtropical gyre
  $V > 0$ in the western boundary $\rightarrow \beta V > 0 \rightarrow R\partial V/\partial x < 0$
- for western boundary layer V decreases to the east $\rightarrow \partial V/\partial x < 0$
- for eastern boundary layer V increases to the east
  $\rightarrow$ no eastern boundary layer
- westward Rossby waves propagate to the west
- they are reflected at the western boundary as short Rossby waves with eastward group velocity
- short Rossby waves are dissipated in the west and form the boundary current



- Schematic of the near-surface circulation (after Schmitz 1996). Subtropical gyres are red, subpolar and polar gyres blue equatorial gyres magenta, Antarctic Circumpolar Current is blue green lines represent exchange between basins and gyres