Dynamische und regionale Ozeanographie WS 2014/15

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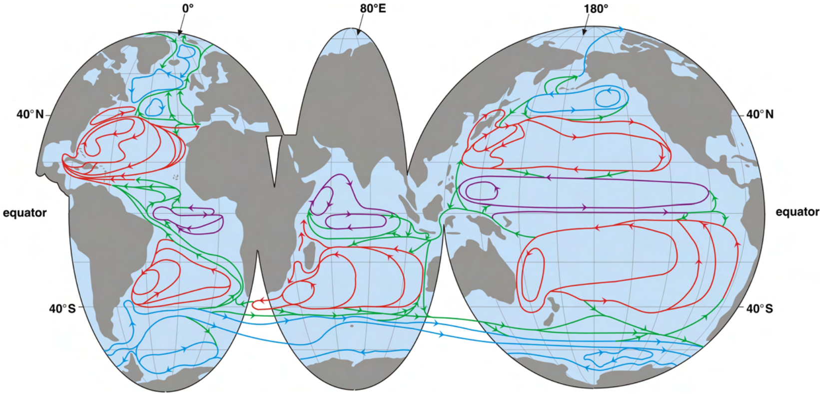
Lecture # 8

Recapitulation
- Ekman layer
- Ekman spirals

Wind driven circulation
- Elementarstromsystem
- Ekman transport
- Ekman pumping
- Sverdrup transport
- Sverdrup meets Ekman



- Schematic of the near-surface circulation (after Schmitz 1996). Subtropical gyres are red, subpolar and polar gyres blue equatorial gyres magenta, Antarctic Circumpolar Current is blue green lines represent exchange between basins and gyres

- planetary geostrophic approximation: $\delta \to 1$, $Ro \to 0$ but finite Ek
- momentum equation becomes

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \text{friction} \quad , \quad fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \text{friction}$$

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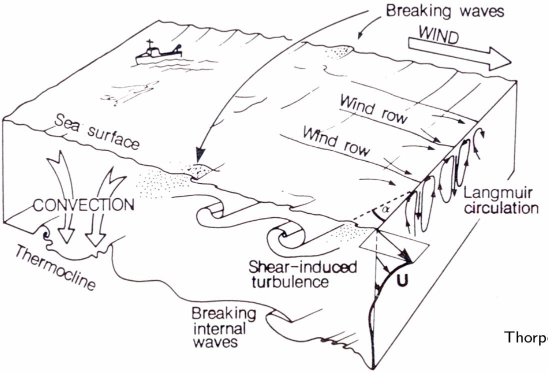
- friction in planetary approximation has not much to do with molecular friction $\to$ assumed scales L and H are too large
- but (non-linear) effects of smaller-scale motions are still present



Thorpe (1985)

- Reynolds-averaged momentum equations become for $Ro \ll 1$

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau^x}{\partial z} \quad , \quad fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{1}{\rho_0}\frac{\partial \tau^y}{\partial z}$$

  with $\tau^x \equiv -\rho_0 \overline{w'u'}$ and $\tau^y = -\rho_0 \overline{w'v'}$
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- stress in the interior is due to small scale turbulence and we expect the stress to decay rapidly over depth $\rightarrow$ Ekman layer depth
- use down-gradient parameterization for stress vector

$$\frac{1}{\rho_0}\tau^x = A_v \frac{\partial u}{\partial z} \quad , \quad \frac{1}{\rho_0}\tau^y = A_v \frac{\partial v}{\partial z}$$

with turbulent vertical viscosity A_v

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  with turbulent vertical viscosity A_v
- A_v depends on turbulence, can be negative or non-existent typical values are $A_v \sim 0.1\ \mathrm{m^2 s^{-1}}$ near the surface mixed layer and much smaller $A_v \sim 10^{-4}\ \mathrm{m^2 s^{-1}}$ in the interior

- 'bulk formulae' for wind stress

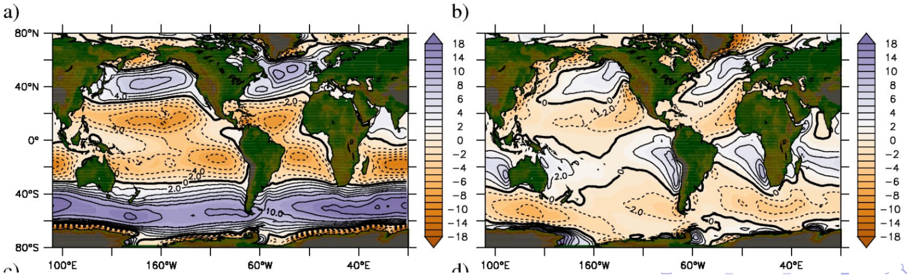
$$\boldsymbol{\tau}^a = \rho_{air} C_D |\boldsymbol{u}_{air} - \boldsymbol{u}_s|(\boldsymbol{u}_{air} - \boldsymbol{u}_s)$$

with air density ρ_{air} and velocity $\boldsymbol{u}_{air}$ and surface ocean velocity $\boldsymbol{u}_s$ and 'drag coefficient' $C_D \approx 1.2 \times 10^{-3}$

- for $|\boldsymbol{u}_s| \ll |\boldsymbol{u}_{air}|$

$$\boldsymbol{\tau}^a = \rho_{air} C_D |\boldsymbol{u}_{air}| \boldsymbol{u}_{air}$$

- zonal (a,c) and meridional component (b,d) of $\boldsymbol{\tau}^a$ in $10^{-2}\,\mathrm{N/m^2}$



- assume $A_v = const$ (which is a special case) and $\nabla_h p = 0$
- momentum equation becomes

$$-fv = -\cancel{\frac{1}{\rho_0}\frac{\partial p}{\partial x}} + A_v \frac{\partial^2 u}{\partial z^2} \quad , \quad fu = -\cancel{\frac{1}{\rho_0}\frac{\partial p}{\partial y}} + A_v \frac{\partial^2 v}{\partial z^2}$$

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- general solution for $f > 0$

$$u + iv = \alpha_+ \exp[(i+1)z/D] + \alpha_- \exp[-(i+1)z/D]$$

where $D = \sqrt{2A_v/|f|}$ is the Ekman layer depth

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  $\rightarrow$ surface Ekman spiral/layer and bottom Ekman spiral/layer

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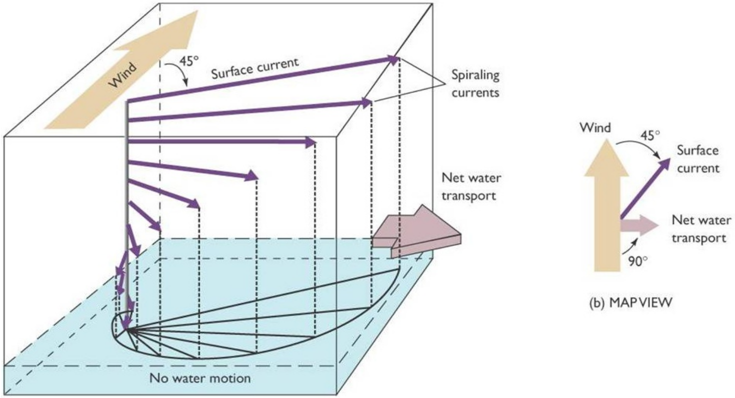
- surface Ekman spiral (for $f > 0$ and $A_v = const$)
  maximum at $z = 0$, decaying and spiraling with depth

$$\boldsymbol{u} = D/(2A_v)e^{z/D}\left((\boldsymbol{\tau}_a - \boldsymbol{k} \times \boldsymbol{\tau}_a)\cos(z/D) + (\boldsymbol{\tau}_a + \boldsymbol{k} \times \boldsymbol{\tau}_a)\sin(z/D)\right)$$

with $\boldsymbol{k} \times \boldsymbol{\tau} = (-\tau^{(y)}, \tau^{(x)}, \not{0})$ (anticlockwise rotation of $\boldsymbol{\tau}$ by 90^o)



- solution for the surface Ekman spiral for $f > 0$ and $A_v = const$

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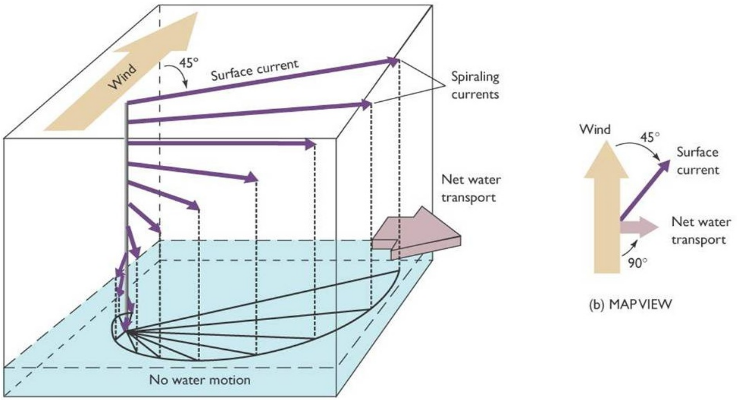

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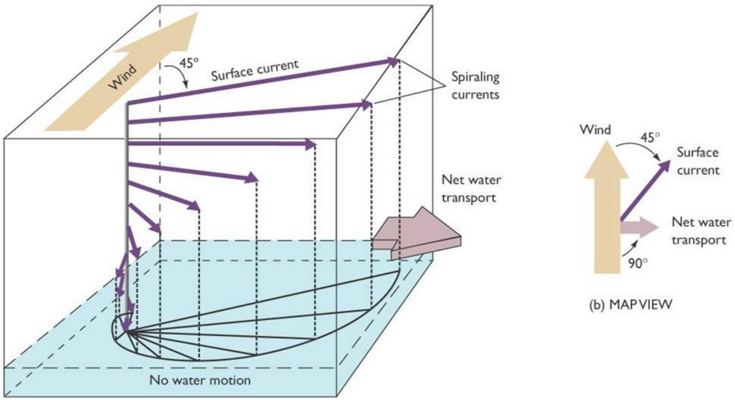
- with $\boldsymbol{u}|_{z=0}$ rotated clockwise ($f > 0$) by 45^o from wind stress $\boldsymbol{\tau}_a$



- solution for the surface Ekman spiral for $f > 0$ and $A_v = const$
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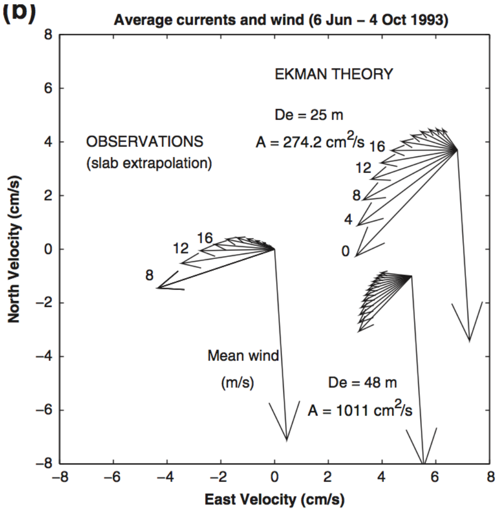
- with $\boldsymbol{u}|_{z=0}$ rotated clockwise $(f > 0)$ by 45^o from wind stress $\boldsymbol{\tau}_a$
- and $\boldsymbol{u}|_{z=-D\pi/2}$ rotated anticlockwise by 45^o from wind stress $\boldsymbol{\tau}_a$ but much smaller

- Ekman-like currents from ADCP in California Current



from Chereskin (1995)

- now include also pressure gradient $\boldsymbol{\nabla}_h p$
- momentum equation in vector form for $Ro \ll 1$

$$f\boldsymbol{k} \times \boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla}_h p + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \quad \text{with} \quad \boldsymbol{k} \times \boldsymbol{u} = (-v, u, \not{0})$$

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- split the flow into geostrophic and frictional (Ekman) components, $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$), governed by

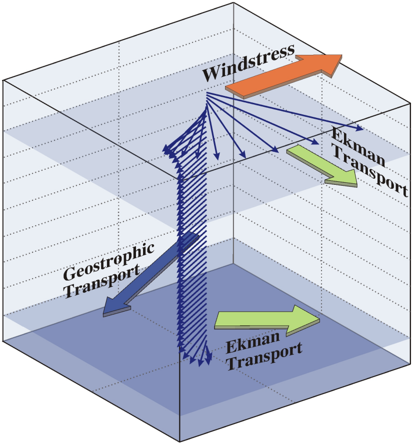
$$f\boldsymbol{k} \times \boldsymbol{u}_G = -\frac{1}{\rho_0}\nabla_h p \qquad \text{and} \qquad f\boldsymbol{k} \times \boldsymbol{u}_E = \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}$$

and the same for continuity equation

$$\nabla \cdot \boldsymbol{u}_G + \frac{\partial w_G}{\partial z} = 0 \qquad \text{and} \qquad \nabla \cdot \boldsymbol{u}_E + \frac{\partial w_E}{\partial z} = 0$$

- sum $\boldsymbol{u}_G + \boldsymbol{u}_E$ satisfies full momentum and continuity equation

- Elementarstromsystem (for $\rho = const$)
- $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$)
  surface and bottom Ekman layers superimposed on geostrophic flow



- vertically integrated velocity

$$\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u}\, dz = \int_{-h}^{0} (\boldsymbol{u}_G + \boldsymbol{u}_E)\, dz = \boldsymbol{U}_G + \boldsymbol{U}_E$$

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- with the (total) transport vector $\boldsymbol{U}$, dimension $\mathrm{m}^2\mathrm{s}^{-1}$

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- transport by the geostrophic velocity $\rightarrow$ geostrophic transport $\boldsymbol{U}_G$
  transport by the Ekman velocity $\rightarrow$ Ekman transport $\boldsymbol{U}_E$

$$f\boldsymbol{k} \times \boldsymbol{u}_E = \frac{1}{\rho_0} \frac{\partial \boldsymbol{\tau}}{\partial z}$$

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$$f\boldsymbol{k} \times \int_{-h}^{0} \boldsymbol{u}_E\, dz = \frac{1}{\rho_0}\left(\boldsymbol{\tau}(z=0) - \boldsymbol{\tau}(z=-h)\right)$$

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with surface wind stress $\boldsymbol{\tau}^a$ and bottom stress $\boldsymbol{\tau}_b$

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  with surface wind stress $\boldsymbol{\tau}^a$ and bottom stress $\boldsymbol{\tau}_b$
- with $\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{U}) = \boldsymbol{k} \times (-V, U, 0) = (-U, -V, 0) = -\boldsymbol{U}$

$$\boldsymbol{U}_E = -\frac{1}{f\rho_0}\boldsymbol{k} \times (\boldsymbol{\tau}^a - \boldsymbol{\tau}_b)$$

- split $\boldsymbol{U}_E$ into surface and bottom Ekman transport

- vertically integrated velocity $\boldsymbol{U} = \boldsymbol{U}_G + \boldsymbol{U}_E$ with geostrophic transport $\boldsymbol{U}_G$ and Ekman transport $\boldsymbol{U}_E$ given by

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  with surface wind stress $\boldsymbol{\tau}^a$ and bottom stress $\boldsymbol{\tau}_b$

- split into surface Ekman transport in surface Ekman layer

$$\boldsymbol{U}_E^{top} = -\frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}^a$$

  orthogonal to wind stress direction (to the right for $f > 0$)

  does not depend on parameterisation of $\boldsymbol{\tau}$ in the interior

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orthogonal to wind stress direction (to the right for $f > 0$)

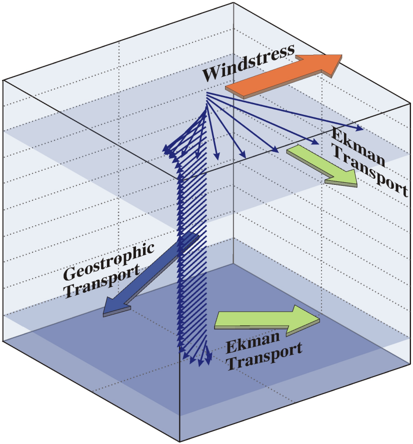
does not depend on parameterisation of $\boldsymbol{\tau}$ in the interior

- and bottom Ekman transport in bottom Ekman layer

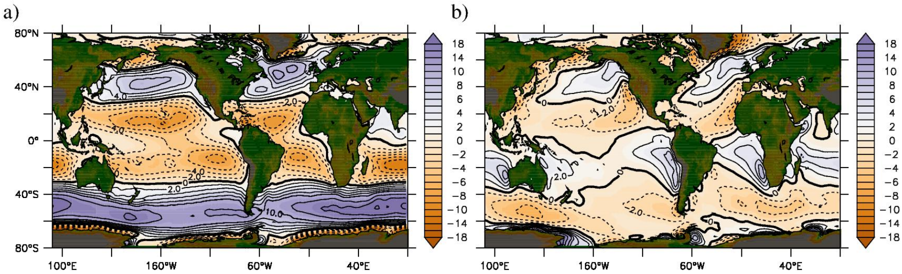
$$\boldsymbol{U}_E^{bot} = \frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}_b$$

depends on parameterisation of $\boldsymbol{\tau}$ in the interior

- Elementarstromsystem
- $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$)
  surface and bottom Ekman layers superimposed on geostrophic flow



- zonal (left) and meridional component (right) of $\boldsymbol{\tau}^a$ in $10^{-2}\,\mathrm{N/m^2}$



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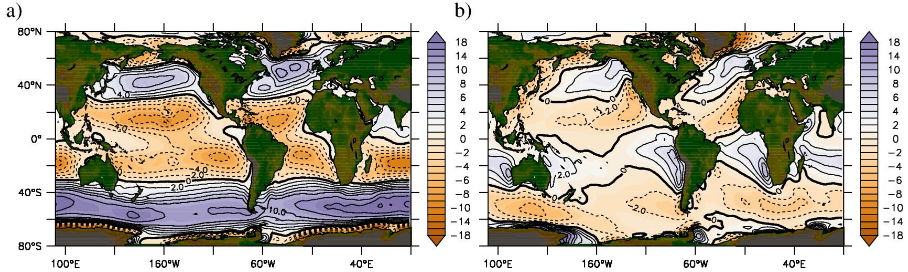

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- equatorward in west wind region poleward in trade wind region

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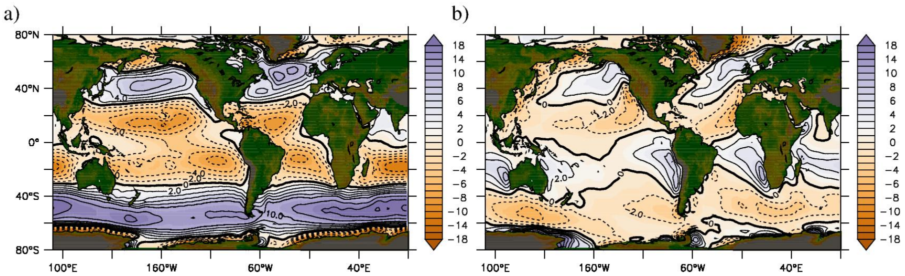

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orthogonal to wind stress direction (to the right for $f > 0$)

- equatorward in west wind region poleward in trade wind region
- convergence between west wind and trade wind region
- divergence at high latitude and at equator

- momentum equation in vector form for $Ro \ll 1$

$$f\boldsymbol{k} \times \boldsymbol{u} = -\frac{1}{\rho_0}\nabla_h p + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \quad \text{with} \quad \boldsymbol{k} \times \boldsymbol{u} = (-v, u, \not{0})$$

- $\boldsymbol{\tau} = (\tau^x, \tau^y)$ is a stress vector with $\boldsymbol{\tau}(z=0) = \boldsymbol{\tau}^a$ where $\boldsymbol{\tau}^a$ is the surface wind stress in $\mathrm{N/m^2}$ acting on the ocean

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- split the flow into geostrophic and frictional (Ekman) components, $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$), governed by

$$f\boldsymbol{k} \times \boldsymbol{u}_G = -\frac{1}{\rho_0}\boldsymbol{\nabla}_h p \qquad \text{and} \qquad f\boldsymbol{k} \times \boldsymbol{u}_E = \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}$$

  and the same for continuity equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_G + \frac{\partial w_G}{\partial z} = 0 \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u}_E + \frac{\partial w_E}{\partial z} = 0$$

- sum $\boldsymbol{u}_G + \boldsymbol{u}_E$ satisfies full momentum and continuity equation

- integrating the continuity equation for $\boldsymbol{u}_E$ and w_E from z to $z = 0$

$$\nabla \cdot \boldsymbol{u}_E + \frac{\partial w_E}{\partial z} = 0$$

yields the vertical Ekman velocity

$$\int_z^0 \nabla \cdot \boldsymbol{u}_E \, dz + \cancel{w_E(z = 0)} - w_E(z) = 0 \ \rightarrow \ w_E(z) = \nabla \cdot \int_z^0 \boldsymbol{u}_E \, dz$$

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$$\boldsymbol{u}_E = D/(2A_v)e^{z/D}\left((\boldsymbol{\tau}_a - \boldsymbol{k} \times \boldsymbol{\tau}_a)\cos(z/D) + (\boldsymbol{\tau}_a + \boldsymbol{k} \times \boldsymbol{\tau}_a)\sin(z/D)\right)$$

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$$w_E|_{z<-D} \approx \nabla \cdot \int_{z<-D}^0 \boldsymbol{u}_E \, dz = \nabla \cdot \boldsymbol{U}_E^{top} = -\nabla \cdot \frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}^a$$

with Ekman pumping $w_E|_{z<-D}$

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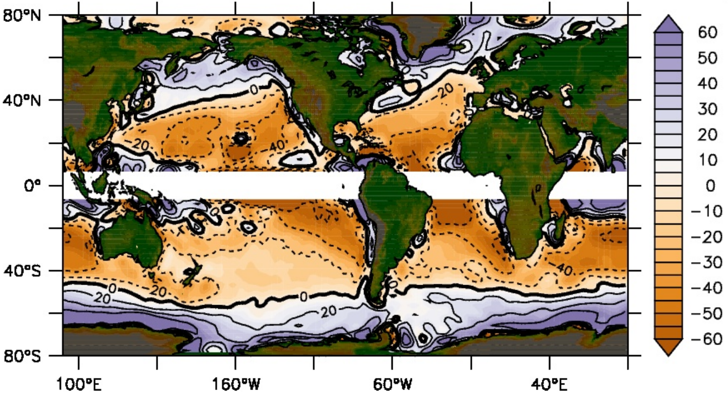
$$\begin{aligned}\boldsymbol{k}\times\nabla\cdot\boldsymbol{\tau} &= \begin{pmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x}\end{pmatrix}\cdot\begin{pmatrix}\tau^{(x)} \\ \tau^{(y)}\end{pmatrix} = -\frac{\partial}{\partial y}\tau^{(x)} + \frac{\partial}{\partial x}\tau^{(y)} \\ &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y}\end{pmatrix}\cdot\begin{pmatrix}\tau^{(y)} \\ -\tau^{(x)}\end{pmatrix} = \nabla\cdot(-\boldsymbol{k}\times\boldsymbol{\tau})\end{aligned}$$

- Ekman pumping w_E in m per year

$$w_E\big|_{z<-D} \approx \boldsymbol{\nabla} \cdot \boldsymbol{U}_E^{top} = \boldsymbol{k} \times \boldsymbol{\nabla} \cdot \frac{\boldsymbol{\tau}^a}{\rho_0 f}$$

with Ekman depth $D \approx 50\,\mathrm{m}$ (depends on A_v)



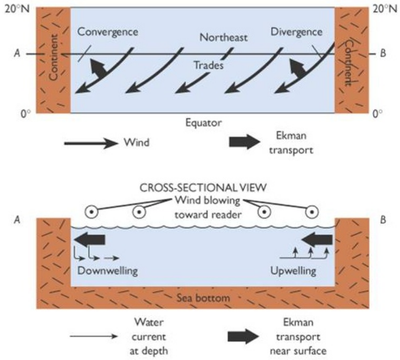
- Ekman transport $\boldsymbol{U}_E^{top}$ and pumping w_E do not depend on A_v

- Ekman pumping w_E

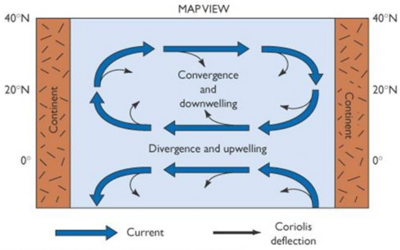
$$w_E\big|_{z<-D} \approx \boldsymbol{\nabla} \cdot \boldsymbol{U}_E^{top}$$

i.e. w_E from divergence of Ekman transport

- $w_E > 0$: Upwelling
  - subpolar gyre
  - at eastern boundaries
  - at equator
- $w_E < 0$: Downwelling
  - subtropical gyres

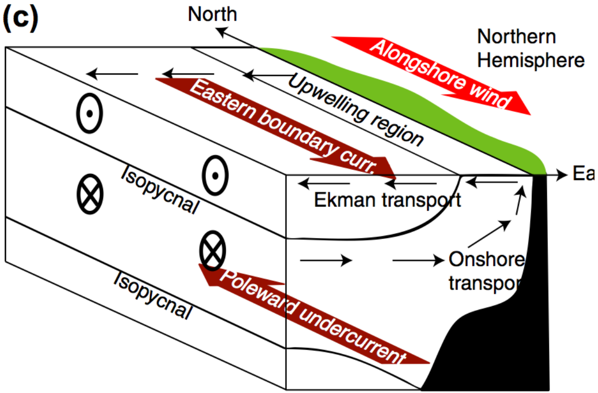


(a) COASTAL DIVERGENCE AND CONVERGENCE



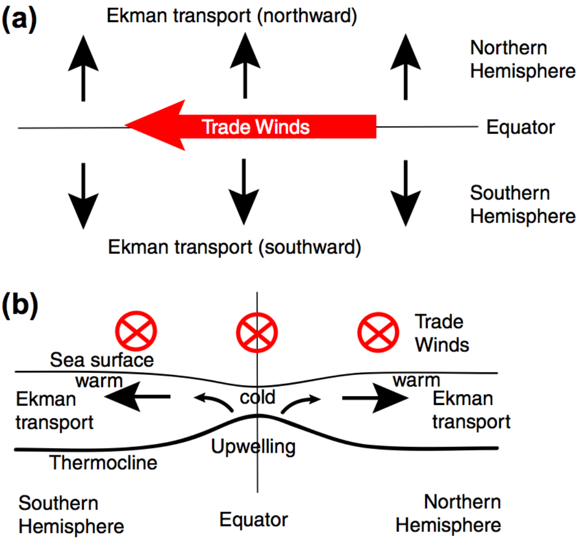
(b) OCEAN DIVERGENCE AND CONVERGENCE

- coastal upwelling

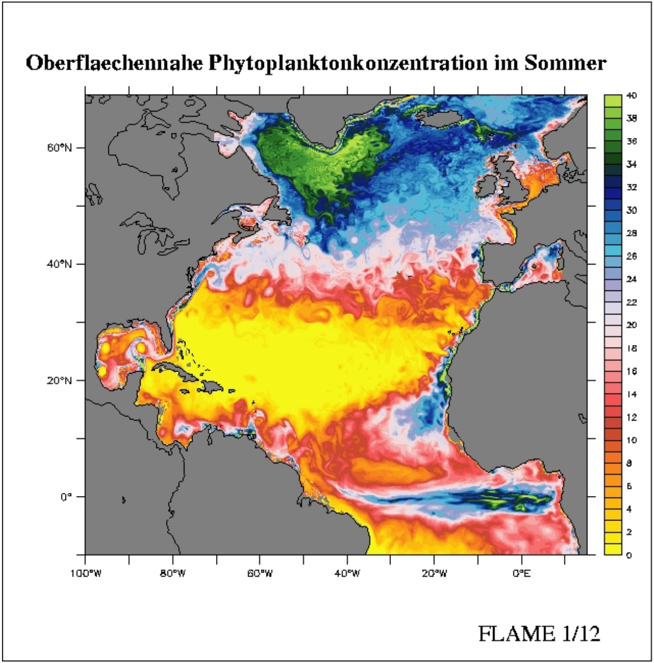


from Talley et al 2011

- equatorial upwelling



from Talley et al 2011


Oberflaechennahe Phytoplanktonkonzentration im Sommer
60°N
40°N
20°N
0°
100°W
80°W
60°W
40°W
20°W
0°E
40
38
36
34
32
30
28
26
24
22
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18
16
14
12
10
8
6
4
2
0
FLAME 1/12

- momentum equation for $Ro \ll 1$

$$f\boldsymbol{k} \times \boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla}_h p + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \quad \text{with} \quad \boldsymbol{k} \times \boldsymbol{u} = (-v, u, 0)$$

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- neglect sea surface height $z = \zeta \rightarrow$ assume rigid lid at $z = 0$
- assume flat bottom at $z = -h = const$
- vertically integrate momentum equation from bottom to surface

$$\rho_0 f\boldsymbol{k} \times \boldsymbol{U} = -\boldsymbol{\nabla}_h \int_{-h}^{0} p dz + \boldsymbol{\tau}_a - \boldsymbol{\tau}_b$$

with transport $\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u} dz$, surface and bottom stress $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$

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- take curl which yields (after a little calculation)

$$\rho_0 \beta V + \rho_0 f \boldsymbol{\nabla}_h \cdot \boldsymbol{U} \quad = \quad \boldsymbol{k} \times \boldsymbol{\nabla} \cdot (\boldsymbol{\tau}_a - \boldsymbol{\tau}_b)$$

with $\beta = df/dy$

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with $\beta = df/dy$

- since $\boldsymbol{\nabla}_h \cdot \boldsymbol{U} = 0$ from continuity equation $\boldsymbol{\nabla}_h \cdot \boldsymbol{u} + \partial w/\partial z = 0$
it follows the famous Sverdrup relation

$$\rho_0 \beta V \quad = \quad \boldsymbol{k} \times \boldsymbol{\nabla} \cdot (\boldsymbol{\tau}_a - \boldsymbol{\tau}_b)$$

- little calculation: curl of vertically integrated momentum equation

$$\rho_0 f \boldsymbol{k} \times \boldsymbol{U} = -\boldsymbol{\nabla}_h \int_{-h}^{0} p dz + \boldsymbol{\tau}_a - \boldsymbol{\tau}_b$$

- little calculation: curl of vertically integrated momentum equation

$$\rho_0 f \boldsymbol{k} \times \boldsymbol{U} = -\boldsymbol{\nabla}_h \int_{-h}^{0} p dz + \boldsymbol{\tau}_a - \boldsymbol{\tau}_b$$

- rewrite component wise

$$-\rho_0 f V = -\frac{\partial}{\partial x} \int_{-h}^{0} p dz + \tau_a^x - \tau_b^x \quad , \quad \rho_0 f U = -\frac{\partial}{\partial y} \int_{-h}^{0} p dz + \tau_a^y - \tau_b^y$$

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- $\partial/\partial y$ of 1. equation minus $\partial/\partial x$ of 2. equation

$$-\frac{\partial}{\partial y}(\rho_0 f V) \quad = \quad -\frac{\partial^2}{\partial x \partial y} \int_{-h}^{0} p dz + \frac{\partial}{\partial y}(\tau_a^x - \tau_b^x)$$

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$$\begin{aligned} -\frac{\partial}{\partial y}(\rho_0 f V) &= -\frac{\partial^2}{\partial x \partial y} \int_{-h}^{0} p dz + \frac{\partial}{\partial y}(\tau_a^x - \tau_b^x) \\ \rho_0 f \frac{\partial U}{\partial x} &= -\frac{\partial^2}{\partial x \partial y} \int_{-h}^{0} p dz + \frac{\partial}{\partial x}(\tau_a^y - \tau_b^y) \end{aligned}$$

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with $\beta = df/dy$ and $\boldsymbol{k} \times \boldsymbol{\nabla} = (-\partial/\partial y, \partial/\partial x, 0)$

- since $\boldsymbol{\nabla}_h \cdot \boldsymbol{U} = 0$ introduce volume transport streamfunction, with

$$U = -\frac{\partial \psi}{\partial y} \quad , \quad V = \frac{\partial \psi}{\partial x} \quad \rightarrow \quad \boldsymbol{U} = \boldsymbol{k} \times \boldsymbol{\nabla} \psi$$

transport $\boldsymbol{U}$ is parallel to contour lines of ψ

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transport $\boldsymbol{U}$ is parallel to contour lines of ψ

- ψ determines transport perpendicular to or "across" section $A \rightarrow B$

$$\int_A^B \boldsymbol{U} \cdot d\boldsymbol{s} = \int_A^B \boldsymbol{k} \times \nabla \psi \cdot d\boldsymbol{s} = \int_A^B \nabla_h \psi \cdot d\boldsymbol{\ell} = \psi(B) - \psi(A)$$

where $d\boldsymbol{s}$ is a line element perpendicular to section $A \rightarrow B$

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$$\rho_0 \beta \, V = \rho_0 \beta \frac{\partial \psi}{\partial x} = \boldsymbol{k} \times \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_a$$

- integration from x to eastern boundary $(x = x_E)$ where $\psi(x_E) = 0$

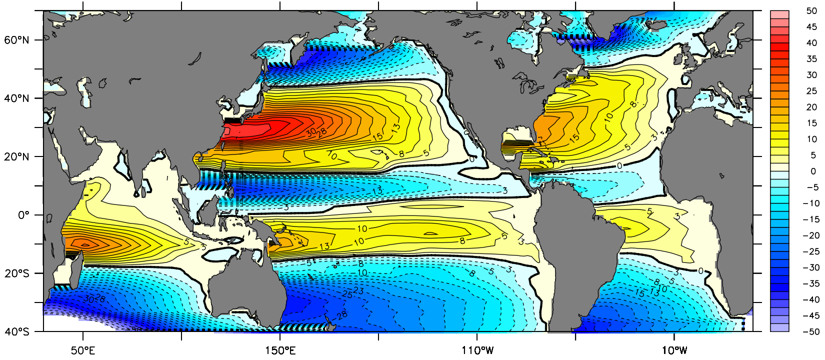
$$\psi(x, y) \quad = \quad -\frac{1}{\rho_0 \beta} \int_x^{x_e} \boldsymbol{k} \times \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_a \, dx$$

  $\rightarrow$ Sverdrup streamfunction

- Sverdrup streamfunction

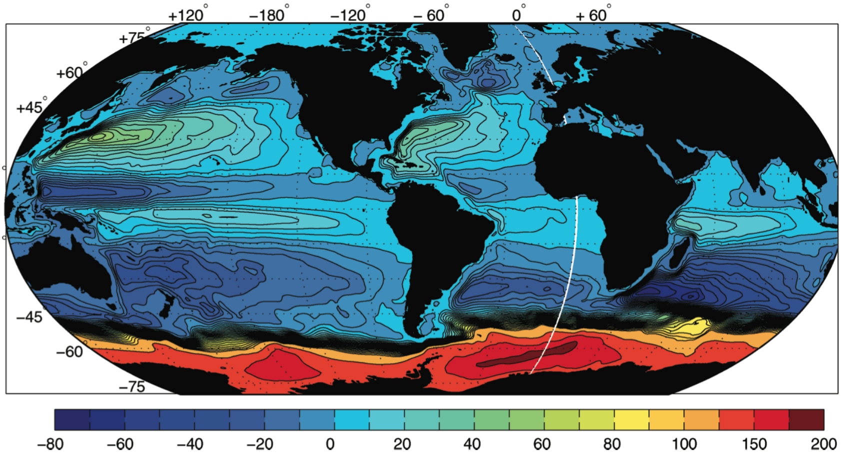
$$\psi = -\frac{1}{\rho_0 \beta} \int_x^{x_e} \boldsymbol{k} \times \boldsymbol{\nabla} \cdot \boldsymbol{\tau}^a \, dx$$

from realistic wind stress in $10^6\,\mathrm{m}^3/\mathrm{s} \equiv 1\,\mathrm{Sv}$

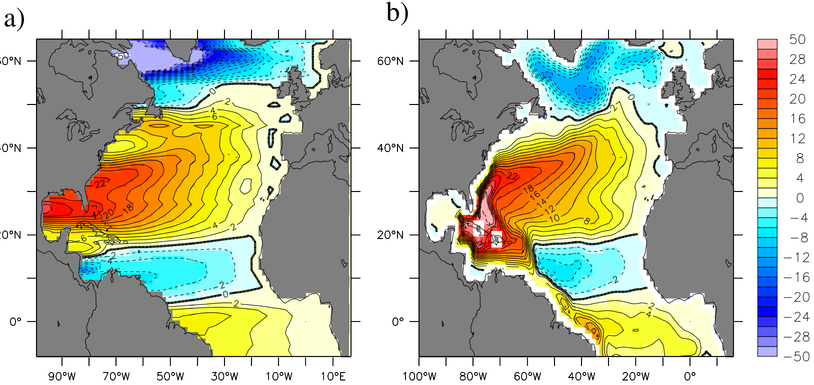


- $\psi(x_E) = 0$ along east eastern boundary but not at western boundary $\rightarrow$ western boundary current not included
- but in the interior ψ is rather realistic

- ψ in a global state estimate in $10^6\,\mathrm{m}^3/\mathrm{s} \equiv 1\,\mathrm{Sv}$



- Streamfunction ψ in *Sv* $= 10^6\,\mathrm{m}^3/\mathrm{s}$ from simple Sverdrup relation
- Streamfunction ψ for a realistic model of the Atlantic Ocean



- simple Sverdrup relation works surprisingly well

- vertically integrated momentum equation

$$
\begin{aligned}
-\rho_0 f V &= -\frac{\partial}{\partial x}\int_{-h}^{0} p dz + \tau_a^x \equiv -\frac{\partial P}{\partial x} + \tau_a^x \\
\rho_0 f U &= -\frac{\partial}{\partial y}\int_{-h}^{0} p dz + \tau_a^y \equiv -\frac{\partial P}{\partial y} + \tau_a^y
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- split in Ekman transport $\boldsymbol{U}_E$ und geostrophic transport $\boldsymbol{U}_G$

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\end{aligned}$$

- but the total transport $\boldsymbol{U} = \boldsymbol{U}_E + \boldsymbol{U}_G$ is non-divergent

$$\boldsymbol{\nabla}_h \cdot \boldsymbol{U} = 0 \quad \rightarrow \quad w_E^{top} = \frac{\beta}{f} V_G$$

Ekman pumping generates southward geostr. transport (for $f > 0$)

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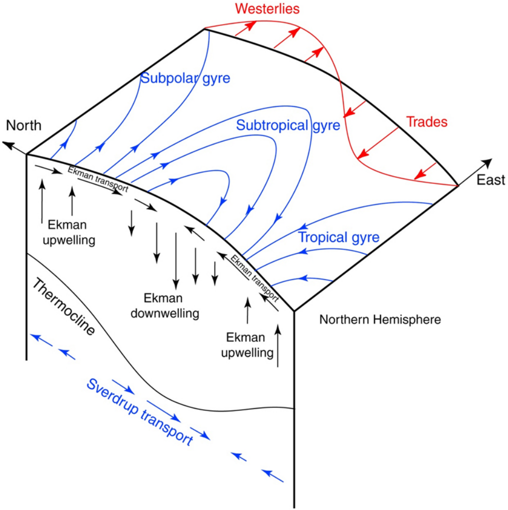

from Talley et al 2011

- Sverdrup relation follows from potential vorticity conservation
- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q \approx \zeta - \frac{f_0}{H} h + f$$

q is conserved for fluid parcels in single layer

- w_E lead to vortex stretching and meridional motion