# Dynamische und regionale Ozeanographie WS 2014/15

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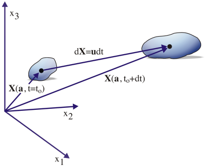
# Lecture # 2

## Recapitulation

- Euler/Lagrange framework
- General conservation equation
- Continuity or mass conservation equation
- Salinity and salt conservation
- Momentum or Navier Stokes equation
- Heat and temperature equation
- Rotating earth

## Approximations and simplifications

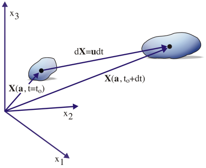
- Boussinesq approximation
- Hydrostatic approximation




- Euler's relation

$$\frac{\delta C}{\delta t} \rightarrow \left(\frac{\partial}{\partial t} C\right)_{parcel} = \frac{\partial}{\partial t} C + \boldsymbol{u} \cdot \boldsymbol{\nabla} C \equiv \frac{D}{Dt} C$$

- $D/Dt = \partial/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$ is often called 'material' or 'substantial' derivative

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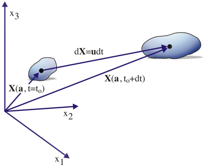


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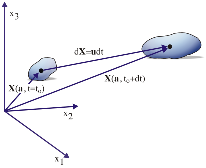


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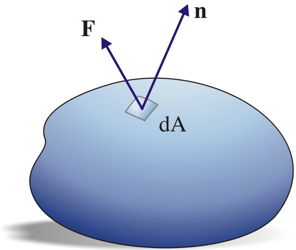
- local rate of change plus change implied by advection of fluid
- if $DC/Dt = 0$, property $C$ of parcels does not change, it's *conservative* (but locally $C$ might change in time)
- Lagrangian frameworks uses left hand side of $DC/Dt$

  Eulerian framework uses right hand side of $DC/Dt$

  both are equivalent but Eulerian framework is often more convenient

- consider volume $V$, *fixed* in space and bounded by a surface $A$
- and a scalar fluid property $C$ concentration
  (in units of $C$ per kg sea water or $\rho C$ in units of $C$ per $m^3$)



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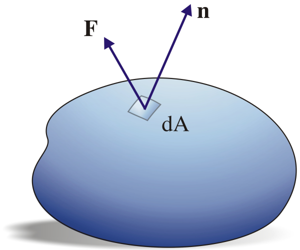
- total amount of the property $C$ in $V$ is given by $\int_V \rho C \, dV$ and may change in time by two ways:



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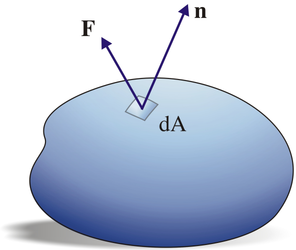

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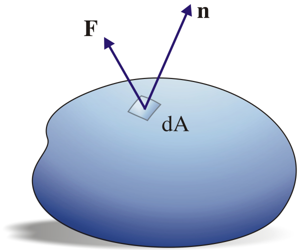
  - by a flux across surface $A$
  - by an interior source or sink
- conservation equation for property concentration $C$

$$\frac{\partial}{\partial t}\rho C = -\boldsymbol{\nabla}\cdot(\rho C\boldsymbol{u} + \boldsymbol{J}) + Q \quad , \quad \rho\frac{DC}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{J} + Q$$

flux form and parcel form



- general conservation law in flux form

$$\frac{\partial}{\partial t}\rho C = -\boldsymbol{\nabla} \cdot (\rho C \boldsymbol{u} + \boldsymbol{J}) + Q$$

- take $C = 1($ kg /kg sea water), $\rightarrow \rho C$ becomes total mass per $m^3$

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- mass conservation or continuity equation

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot\rho\boldsymbol{u}$$

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$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot\rho\boldsymbol{u}$$

- possible to rewrite flux form (above) to parcel form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho \equiv \frac{D\rho}{Dt} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

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- with specific volume $v = 1/\rho$ continuity equation becomes

$$\frac{D\rho}{Dt} = \frac{D}{Dt} v^{-1} = -\frac{1}{v^2}\frac{Dv}{Dt} = -\frac{1}{v}\boldsymbol{\nabla} \cdot \boldsymbol{u}$$

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parcel form of continuity equation

- general conservation law in flux form

$$\frac{\partial}{\partial t}\rho C = -\boldsymbol{\nabla}\cdot(\rho C\boldsymbol{u} + \boldsymbol{J}) + Q$$

- salt conservation equation with $C = S$, $Q = 0$ but $\boldsymbol{J} = \boldsymbol{J}_S$

$$\frac{\partial}{\partial t}\rho S = -\boldsymbol{\nabla}\cdot(\rho S\boldsymbol{u} + \boldsymbol{J}_S)$$

with salt flux $\boldsymbol{J}_S$ by molecular diffusion

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- rewrite to parcel form given by

$$\rho\frac{\partial S}{\partial t} + S\frac{\partial\rho}{\partial t} \quad = \quad -\boldsymbol{\nabla}\cdot\rho S\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{J}_S$$

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\rho\frac{DS}{Dt} &= -\boldsymbol{\nabla}\cdot\boldsymbol{J}_S
\end{aligned}$$

using continuity equation $\partial\rho/\partial t = -\boldsymbol{\nabla}\cdot\rho\boldsymbol{u}$ times $S$

- general conservation equation

$$\frac{\partial}{\partial t}\rho C = -\boldsymbol{\nabla}\cdot(\rho C\boldsymbol{u}+\boldsymbol{J})+Q \ , \ \rho\frac{DC}{Dt}=-\boldsymbol{\nabla}\cdot\boldsymbol{J}+Q$$

flux form and parcel form

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  flux form and parcel form

- flux form for momentum component $u_i = C$

$$\frac{\partial}{\partial t}\rho u_i = -\boldsymbol{\nabla}\cdot\left(\rho u_i\boldsymbol{u}+\boldsymbol{J}^{(i)}\right)+Q_i$$

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$$\rho\frac{Du_i}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{J}^{(i)} + Q_i = -\frac{\partial}{\partial x_j}J_j^{(i)} + Q_i$$

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- in vector form and with (stress) tensor $-\Pi_{ji} = J_j^{(i)}$

$$\rho\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\nabla}\cdot\boldsymbol{\Pi}+\boldsymbol{Q}$$

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$$\rho\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\nabla}\cdot\boldsymbol{\Pi}+\boldsymbol{Q}$$

- $\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}$ and $\boldsymbol{Q}$ are forces (per volume) acting on the water parcel

- balance of momentum in parcel form

$$\rho \frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} + \boldsymbol{f}^v = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} + \boldsymbol{f}^v$$

with the (mechanical) *pressure* as the mean normal inward stress

$$p = -\frac{1}{3}\left(\Pi_{11} + \Pi_{22} + \Pi_{33}\right) = -\frac{1}{3}\Pi_{ii} = -\frac{1}{3}\mathrm{tr}\,\boldsymbol{\Pi}$$

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- Newtonian fluid: relation between friction and velocity shear

$$\Sigma_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_\ell}{\partial x_\ell}\delta_{ij}\right)$$

with the (dynamical) viscosity $\nu$

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with the (dynamical) viscosity $\nu$

- Navier-Stokes equation for Newtonian fluid (and constant $\nu$)

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} + \frac{\nu}{3} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \boldsymbol{f}^v$$

- conservation equation for in situ temperature $T$

$$\rho c_p \frac{DT}{Dt} = \alpha T \frac{Dp}{Dt} + \frac{\partial H}{\partial S} \boldsymbol{\nabla} \cdot \boldsymbol{J}_S - \boldsymbol{\nabla} \cdot \boldsymbol{J}_H + \rho\epsilon$$

with enthalpy $H$, specific heat $c_p = \partial H / \partial T$, thermal expansion coefficient $\alpha = -1/\rho\, \partial\rho / \partial T$, kinetic energy dissipation $\rho\epsilon = \Sigma_{ij}^2$ and molecular diffusive enthalpy flux $\boldsymbol{J}_H$

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- assume adiabatic conditions, i.e. $\boldsymbol{J}_S = 0$, $\boldsymbol{J}_H = 0$ and $\epsilon = 0$

$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{Dp}{Dt} = 0$$

- conservation equation for in situ temperature $T$

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$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{Dp}{Dt} = 0 \quad \text{or} \quad \frac{DT}{Dt} = \Gamma \frac{Dp}{Dt}$$

with adiabatic lapse rate $\Gamma = \alpha T / (\rho c_p)$

- conservation equation for in situ temperature $T$

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- in situ temperature is not "conserved"

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- in situ temperature is not "conserved"
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- adiabatic lapse rate can also be defined by $\Gamma = (\partial T / \partial p)_{ad}$

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- in situ temperature is not "conserved"
- changes in temperature and pressure are related by $dT = \Gamma dp$
- adiabatic lapse rate can also be defined by $\Gamma = (\partial T/\partial p)_{ad}$
- typical value is $\Gamma \approx 10^{-8}\,\mathrm{K/Pa} = 10^{-4}\,\mathrm{K/dbar} \sim 0.1\,\mathrm{K/km}$

- use equation for "conservative temperature" $\Theta$ instead

$$\rho\frac{D\Theta}{Dt} = \left(\frac{\theta}{T}\frac{\partial H}{\partial S} - \frac{\partial H^0}{\partial S}\right)\boldsymbol{\nabla}\cdot\frac{\boldsymbol{J}_S}{c_p^\star} + \frac{\theta}{T}\left(-\boldsymbol{\nabla}\cdot\frac{\boldsymbol{J}_H}{c_p^\star} + \rho\frac{\epsilon}{c_p^\star}\right)$$

with "potential enthalpy" $H^0 = H(p = p_{ref})$, reference specific heat $c_p^\star = const$, "potential temperature" $\theta = \partial H^0/\partial\eta$ (with entropy $\eta$)

- now assume $\theta/T \approx 1$ with relative error of $10^{-3}$

- use equation for "conservative temperature" $\Theta$ instead

$$\rho \frac{D\Theta}{Dt} = \left( \frac{\theta}{T} \frac{\partial H}{\partial S} - \frac{\partial H^0}{\partial S} \right) \boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_S}{c_p^\star} + \frac{\theta}{T} \left( -\boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_H}{c_p^\star} + \rho \frac{\epsilon}{c_p^\star} \right)$$

with "potential enthalpy" $H^0 = H(p = p_{ref})$, reference specific heat $c_p^\star = const$, "potential temperature" $\theta = \partial H^0 / \partial \eta$ (with entropy $\eta$)

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- now assume $\theta / T \approx 1$ with relative error of $10^{-3}$
- neglect effect of salt fluxes compared to heat flux term $\boldsymbol{J}_H$
- neglect effect of dissipation compared to heat flux term
- get temperature equation containing the divergence of $\boldsymbol{J}_H$

$$\rho \frac{D\Theta}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term}$$

with $\boldsymbol{J}_\Theta = \boldsymbol{J}_H / c_p^\star$

- a new coordinate system rotates with $\boldsymbol{\Omega}$

$$\frac{D\boldsymbol{X}}{Dt} = \boldsymbol{u} = \boldsymbol{u}^{rot} + \boldsymbol{\Omega} \times \boldsymbol{X}$$

velocity within the rotating frame is $\boldsymbol{u}^{rot}$

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velocity within the rotating frame is $\boldsymbol{u}^{rot} = (D/Dt)^{rot}\boldsymbol{X}$

temporal change within the rotating frame is $(D/Dt)^{rot}$

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$$\frac{D\boldsymbol{u}}{Dt} = \frac{D\boldsymbol{u}^{rot}}{Dt} + \boldsymbol{\Omega} \times \frac{D\boldsymbol{X}}{Dt}$$

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- $D\boldsymbol{u}/Dt$ is the acceleration of the fluid element in the absolute frame $(D/Dt)^{rot}\boldsymbol{u}^{rot}$ is the acceleration measured by a co-rotating observer
- two additional (apparent) forces: Coriolis and centrifugal force

- momentum equation in rotating frame

$$
\begin{aligned}
\rho\frac{D\boldsymbol{u}}{Dt} &= \rho\left(\frac{D}{Dt}\right)^{rot}\boldsymbol{u}^{rot} + 2\rho\boldsymbol{\Omega}\times\boldsymbol{u}^{rot} + \rho\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{X}) \\
&= -\boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot\boldsymbol{\Sigma} + \boldsymbol{f}^{v}
\end{aligned}
$$

- momentum equation in rotating frame

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \left(\frac{D}{Dt}\right)^{rot} \boldsymbol{u}^{rot} + 2\rho \boldsymbol{\Omega} \times \boldsymbol{u}^{rot} + \rho \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})$$
$$= -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} + \boldsymbol{f}^{v}$$

- or, dropping the index $^{rot}$ from now on

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho \boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} + \boldsymbol{f}^{v} - \rho \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})$$

- momentum equation in rotating frame

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \left(\frac{D}{Dt}\right)^{rot} \boldsymbol{u}^{rot} + 2\rho\boldsymbol{\Omega} \times \boldsymbol{u}^{rot} + \rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})$$
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- with the Coriolis force $-2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$

- momentum equation in rotating frame

$$\begin{aligned} \rho \frac{D\boldsymbol{u}}{Dt} &= \rho \left(\frac{D}{Dt}\right)^{rot} \boldsymbol{u}^{rot} + 2\rho\boldsymbol{\Omega} \times \boldsymbol{u}^{rot} + \rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X}) \\ &= -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} + \boldsymbol{f}^{v} \end{aligned}$$

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- with the Coriolis force $-2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$
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- momentum equation in rotating frame

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \left(\frac{D}{Dt}\right)^{rot} \boldsymbol{u}^{rot} + 2\rho\boldsymbol{\Omega} \times \boldsymbol{u}^{rot} + \rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})$$
$$= -\nabla p + \nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{f}^{v}$$

- or, dropping the index $^{rot}$ from now on

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p - 2\rho\boldsymbol{\Omega} \times \boldsymbol{u} + \nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{f}^{v} - \rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})$$

- with the Coriolis force $-2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$
- and the centrifugal force $-\rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})$
- The combined force in the equation of motion is given by

$$-\rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X}) - \rho\nabla\Phi_E = -\rho\nabla\Phi$$

## Summary conservation laws

- momentum equation

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho \boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} \; - \rho \boldsymbol{\nabla}(\Phi + \Phi_{tide})$$

with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- continuity equation

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u} \quad , \quad \rho \frac{Dv}{Dt} = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

- salt conservation equation

$$\rho \frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S$$

- conservative temperature equation

$$\rho \frac{D\Theta}{Dt} \quad = \quad -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term}$$

- equation of state with conservative temperature as state variable

$$\rho \; = \; \rho(S, \Theta, p)$$

- Continuity equation or conservation of mass

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})$$

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- introduce scaled variables with primes

$$\rho = \rho_0 \rho \quad , \quad \boldsymbol{u} = U\boldsymbol{u}' \quad , \quad \boldsymbol{x} = L\boldsymbol{x}' \quad , \quad t = Tt'$$

with the dimensionless functions $\rho'$, $\boldsymbol{u}'$, etc of $O(1)$
with constants $\rho_0$, $U$ taking dimensions and magnitudes
and with $\partial/\partial t = (1/T)\partial/\partial t'$, and $\partial/\partial x = (1/L)\partial/\partial x'$, etc

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- this yields

$$\begin{aligned} \frac{\rho_0}{T}\frac{\partial \rho'}{\partial t'} &= -\frac{\rho_0 U}{L}\boldsymbol{\nabla}' \cdot (\rho' \boldsymbol{u}') \\ \frac{\partial \rho'}{\partial t'} &= -\frac{UT}{L}\boldsymbol{\nabla}' \cdot (\rho' \boldsymbol{u}') \end{aligned}$$

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- now forget all primes

$$\frac{\partial \rho}{\partial t} = -\frac{UT}{L} \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})$$

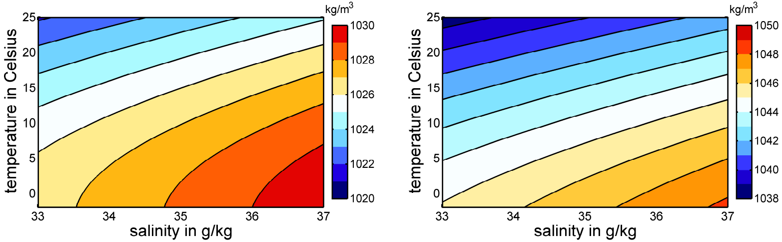
- equation of state for seawater

$$\rho = \rho(S, T, p)$$

function of salinity $S$, temperature $T$ and pressure $p$

- no analytical (exact) expressions for the function $\rho(S, T, p)$
- empirical expressions with relative accuracy of $(3 - 5) \times 10^{-6}$
  $\rightarrow$ TEOS: http://www.teos-10.org/

$\rho(S, T, p = 0\,\mathrm{Pa})$ $\rho(S, T, p = 1000\,\mathrm{dbar} = 10^7\,\mathrm{Pa})$



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  with a large mean value $\rho_0 = 1000\,\mathrm{kg/m^3}$
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- this yields in the continuity equation

$$\frac{1}{T}\frac{\partial}{\partial t'}(\rho_0 + \varrho\rho') \quad = \quad -\frac{U}{L}\boldsymbol{\nabla}' \cdot ((\rho_0 + \varrho\rho')\boldsymbol{u}')$$

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- since $\rho_0/\varrho \gg 1$ and rest of $O(1)$ (as long as $UT/L \geq O(1)$ )
  it follows that

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \;\rightarrow\; \boldsymbol{\nabla} \cdot \boldsymbol{u} \approx 0 \;: \quad \text{Boussinesq approximation}$$

  mass conservation is replaced by volume conservation

- momentum equation (without friction and tides)

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\nabla} \Phi$$

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- using $\rho = \rho_0 + \varrho\rho'$ yields

$$(\rho_0 + \varrho\rho') \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2(\rho_0 + \varrho\rho') \boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho') \boldsymbol{\nabla} \phi$$

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- consider 3. component for $\boldsymbol{u} = 0$ and $\phi = gz$

$$\frac{\partial p}{\partial z} = -(\rho_0 + \varrho\rho')g \approx -\rho_0 g$$

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- motivates to set $p \equiv p_0(z) + p'$ with $p_0 \gg p'$ and to set

$$\frac{\partial p_0}{\partial z} \equiv -\rho_0 g$$

hydrostatic balance of $p_0$ with constant density $\rho_0$

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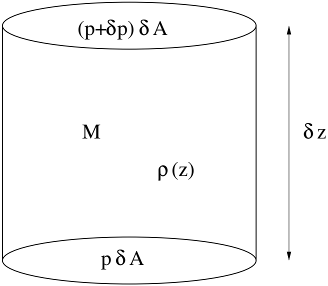
$$\frac{\partial p_0}{\partial z} \equiv -\rho_0 g$$

  hydrostatic balance of $p_0$ with constant density $\rho_0$

- momentum equation becomes

$$(\rho_0 + \varrho\rho')\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}(p_0(z) + p') - 2(\rho_0 + \varrho\rho')\boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho')\boldsymbol{\nabla}\phi$$

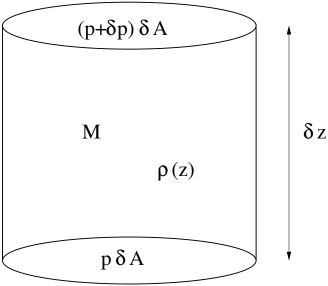
- consider a column of the ocean




- pressure force at bottom

$$F_B = p\delta A$$

- consider a column of the ocean




- pressure force at bottom

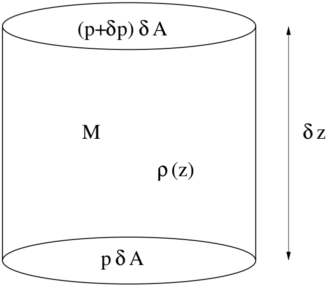
$$F_B = p\delta A$$

- pressure force at top

$$F_T = -(p + \delta p)\delta A$$

(positive upward)

- consider a column of the ocean

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$$F_T = -(p + \delta p)\delta A \approx -(p + \frac{\partial p}{\partial z}\delta z)\delta A$$

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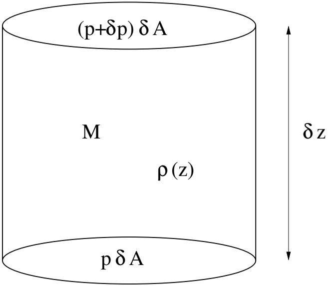


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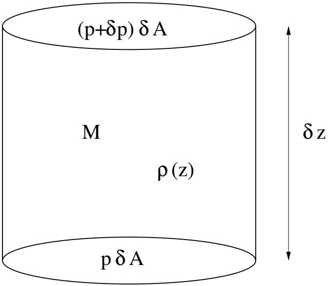
(positive upward)

- mass of cylinder is $M = \rho\delta A\delta z$ and gravity force is $F_g = -gM$

- consider a column of the ocean

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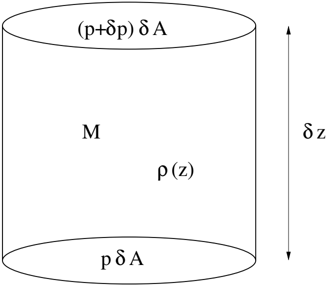
(positive upward)

- mass of cylinder is $M = \rho\delta A\delta z$ and gravity force is $F_g = -gM$
- if no other forces act and column does not accelerate

$$F_B + F_T + F_g = 0$$

- consider a column of the ocean

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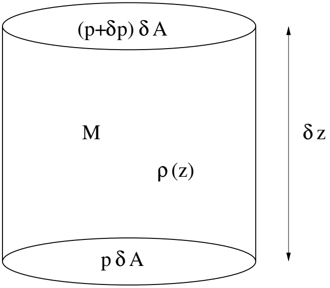
$$F_B + F_T + F_g = 0 \;\rightarrow\; p\delta A - (p + \frac{\partial p}{\partial z}\delta z)\delta A - g\rho\delta A\delta z = 0$$

- consider a column of the ocean

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- mass of cylinder is $M = \rho\delta A\delta z$ and gravity force is $F_g = -gM$
- if no other forces act and column does not accelerate

$$F_B + F_T + F_g = 0 \ \rightarrow \ p\delta A - (p + \frac{\partial p}{\partial z}\delta z)\delta A - g\rho\delta A\delta z = 0$$

or

$$-\frac{\partial p}{\partial z} - g\rho = 0 \ \rightarrow \ \text{hydrostatic balance}$$

- momentum equation becomes

$$(\rho_0 + \varrho\rho')\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}(p_0(z) + p') - 2(\rho_0 + \varrho\rho')\boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho')\boldsymbol{\nabla}\phi$$

- momentum equation becomes

$$(\rho_0 + \varrho\rho')\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}(p_0(z) + p') - 2(\rho_0 + \varrho\rho')\boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho')\boldsymbol{\nabla}\phi$$

- since by construction

$$-\frac{\partial p_0}{\partial z} - g\rho_0 = 0 \quad \text{or} \quad -\boldsymbol{\nabla} p_0(z) - \rho_0\boldsymbol{\nabla}\phi = 0$$

background gravity and vertical pressure gradient completely drop

- momentum equation becomes

$$(\rho_0 + \varrho\rho')\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}(p_0(z) + p') - 2(\rho_0 + \varrho\rho')\boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho')\boldsymbol{\nabla}\phi$$

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background gravity and vertical pressure gradient completely drop

- since $\rho_0 \gg \varrho$ momentum equation further simplifies to

$$\rho_0\frac{D\boldsymbol{u}}{Dt} \approx -\boldsymbol{\nabla} p' - 2\rho_0\boldsymbol{\Omega} \times \boldsymbol{u} - \varrho\rho'\boldsymbol{\nabla}\phi$$

finally set $p' \to p$ and $\varrho\rho' \to \rho$

but remember that pressure $p$ and density $\rho$ are now *perturbations*

- momentum equation becomes

$$(\rho_0 + \varrho\rho')\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}(p_0(z) + p') - 2(\rho_0 + \varrho\rho')\boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho')\boldsymbol{\nabla}\phi$$

- since by construction

$$-\frac{\partial p_0}{\partial z} - g\rho_0 = 0 \quad \text{or} \quad -\boldsymbol{\nabla} p_0(z) - \rho_0\boldsymbol{\nabla}\phi = 0$$

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$$\rho_0\frac{D\boldsymbol{u}}{Dt} \approx -\boldsymbol{\nabla} p' - 2\rho_0\boldsymbol{\Omega} \times \boldsymbol{u} - \varrho\rho'\boldsymbol{\nabla}\phi$$

finally set $p' \to p$ and $\varrho\rho' \to \rho$

but remember that pressure $p$ and density $\rho$ are now *perturbations*

- salinity equation with $\rho = \rho_0 + \varrho\rho'$ becomes

$$(\rho_0 + \varrho\rho')\frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S \quad \to \quad \rho_0\frac{DS}{Dt} \approx -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S$$

and similar for temperature

## Summary exact conservation laws

- momentum equation

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho \boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} \; - \rho \boldsymbol{\nabla}(\Phi + \Phi_{tide})$$

with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- continuity equation

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u} \quad , \quad \rho \frac{Dv}{Dt} = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

- salt conservation equation

$$\rho \frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S$$

- conservative temperature equation

$$\rho \frac{D\Theta}{Dt} \quad = \quad -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term}$$

- equation of state with conservative temperature as state variable

$$\rho \; = \; \rho(S, \Theta, p)$$

# Conservation laws in Boussinesq approximation

- momentum equation

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} \ - \rho \boldsymbol{\nabla}(\Phi + \Phi_{tide})$$

with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- continuity equation

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

- salt conservation equation

$$\rho_0 \frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S$$

- conservative temperature equation

$$\rho_0 \frac{D\Theta}{Dt} \quad = \quad -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term}$$

- equation of state with conservative temperature as state variable

$$\rho \ = \ \rho(S, \Theta, p_0)$$

Recapitulation
- Euler/Lagrange framework
- General conservation equation
- Continuity or mass conservation equation
- Salinity and salt conservation
- Momentum or Navier Stokes equation
- Heat and temperature equation
- Rotating earth

Approximations and simplifications
- Boussinesq approximation
- Hydrostatic approximation

- scale continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{U}{L}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \frac{W}{H}\frac{\partial w'}{\partial z'} = 0$$

with lateral scale $L$ and vertical scale $H$

and lateral velocity scale $U$ and vertical velocity scale $W$

- scale continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \rightarrow \quad \frac{U}{L}\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}\right)+\frac{W}{H}\frac{\partial w'}{\partial z'}=0$$

  with lateral scale $L$ and vertical scale $H$

  and lateral velocity scale $U$ and vertical velocity scale $W$

- for $W/H \gg U/L$ it follows that $\partial w/\partial z = 0$

  such that $w = 0$ considering bottom or top boundaries

  $\rightarrow$ scaling becomes inconsistent

- scale continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{U}{L}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \frac{W}{H}\frac{\partial w'}{\partial z'} = 0$$

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  $\rightarrow$ scaling becomes inconsistent

- only cases $W/H \sim U/L$ or $W/H \ll U/L$ are possible

- scale continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{U}{L}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \frac{W}{H}\frac{\partial w'}{\partial z'} = 0$$

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  $\rightarrow$ scaling becomes inconsistent

- only cases $W/H \sim U/L$ or $W/H \ll U/L$ are possible

- now define aspect ratio $\delta = H/L$ ("deepness" of the flow) with

$$W = UH/L = \delta U$$

  which means that for $\delta \sim 1 \;\rightarrow\; W \sim U$ and for $\delta \ll 1 \;\rightarrow\; W \ll U$

- scale continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \rightarrow \quad \frac{U}{L}\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}\right)+\frac{W}{H}\frac{\partial w'}{\partial z'}=0$$

with lateral scale $L$ and vertical scale $H$

and lateral velocity scale $U$ and vertical velocity scale $W$

- for $W/H \gg U/L$ it follows that $\partial w/\partial z = 0$

such that $w = 0$ considering bottom or top boundaries

$\rightarrow$ scaling becomes inconsistent

- only cases $W/H \sim U/L$ or $W/H \ll U/L$ are possible

- now define aspect ratio $\delta = H/L$ ("deepness" of the flow) with

$$W = UH/L = \delta U$$

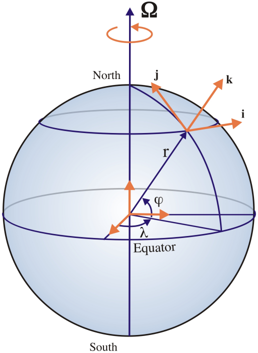
which means that for $\delta \sim 1 \;\rightarrow\; W \sim U$ and for $\delta \ll 1 \;\rightarrow\; W \ll U$

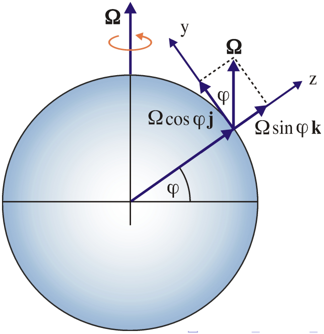
- since $\delta \ll 1$ for large-scale flow in the ocean $W \ll U$

i.e. shallow water yields small (but still important!) $w$

- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$



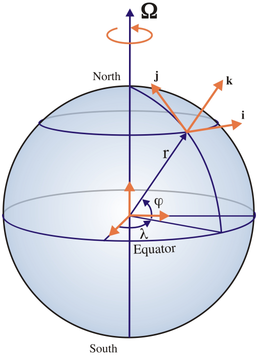



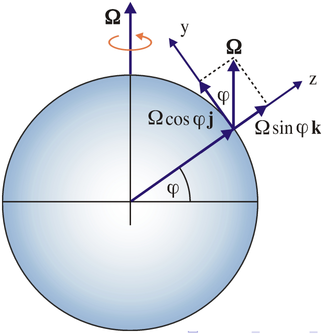
- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

- with $\boldsymbol{\Omega} = (0, \Omega \cos\varphi, \Omega \sin\varphi)$

$$\boldsymbol{\Omega} \times \boldsymbol{u} = \Omega \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \Omega \begin{pmatrix} w\cos\varphi - v\sin\varphi \\ u\sin\varphi \\ -u\cos\varphi \end{pmatrix}$$



- aspect ratio $\delta = H/L \ll 1$ , $W = \delta U$
- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

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- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

- consider first momentum equation

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} - 2\Omega(w \cos\varphi - v \sin\varphi)$$

- aspect ratio $\delta = H/L \ll 1$ , $W = \delta U$
- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

- consider first momentum equation

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} - 2\Omega(w \cos\varphi - v \sin\varphi)$$

- scaling of each term yields

$$\frac{U}{T} \;,\; \left(\frac{U^2}{L}, \frac{WU}{H}\right) \;\sim\; \frac{P}{\rho_0 L} \;,\; \Omega W \;,\; \Omega U$$

divide by $\Omega U$ to get magnitudes relative to (vertical) Coriolis force

$$\frac{1}{T\Omega} \;,\; \frac{U}{L\Omega} \;\sim\; \frac{P}{\rho_0 L \Omega U} \;,\; \delta \ll 1 \;,\; 1$$

- aspect ratio $\delta = H/L \ll 1 \quad , \quad W = \delta U$
- consider first momentum equation

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} - 2\Omega(w \cos\varphi - v \sin\varphi)$$

- scaling yields

$$\frac{1}{T\Omega} \quad , \quad \frac{U}{L\Omega} \quad \sim \quad \frac{P}{\rho_0 L \Omega U} \quad , \quad \delta \ll 1 \quad , \quad 1$$

- aspect ratio $\delta = H/L \ll 1 \quad , \quad W = \delta U$
- consider first momentum equation

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} - 2\Omega(w \cos\varphi - v \sin\varphi)$$

- scaling yields

$$\frac{1}{T\Omega} \quad , \quad \frac{U}{L\Omega} \quad \sim \quad \frac{P}{\rho_0 L \Omega U} \quad , \quad \delta \ll 1 \quad , \quad 1$$

- set $T = L/U$ and define Rossby number $Ro = U/(L\Omega)$

$$Ro \quad , \quad Ro \quad \sim \quad \frac{P}{\rho_0 L \Omega U} \quad , \quad \delta \ll 1 \quad , \quad 1$$

- $Ro$ compares momentum advection with Coriolis force

  for large-scale flow in the ocean $Ro \leq 1$

- aspect ratio $\delta = H/L \ll 1 \quad , \quad W = \delta U$
- consider first momentum equation

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - 2\Omega(w \cos\varphi - v \sin\varphi)$$

- scaling yields

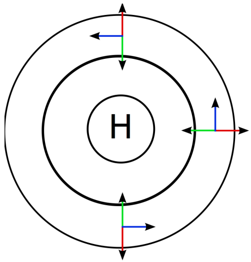
$$\frac{1}{T\Omega} \quad , \quad \frac{U}{L\Omega} \quad \sim \quad \frac{P}{\rho_0 L\Omega U} \quad , \quad \delta \ll 1 \quad , \quad 1$$

- set $T = L/U$ and define Rossby number $Ro = U/(L\Omega)$

$$Ro \quad , \quad Ro \quad \sim \quad \frac{P}{\rho_0 L\Omega U} \quad , \quad \delta \ll 1 \quad , \quad 1$$

- $Ro$ compares momentum advection with Coriolis force
  for large-scale flow in the ocean $Ro \leq 1$
- assume dominant geostrophic balance: $P/(\rho_0 L\Omega U) = 1$
  but still keep terms of $O(Ro)$ (but not those of $O(\delta)$)!

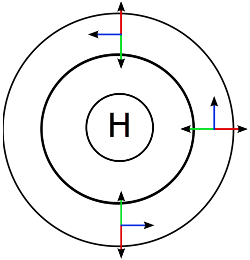
- geostrophy: exact balance between pressure and Coriolis force




$$
\begin{aligned}
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\Omega v \sin\varphi \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\Omega u \sin\varphi
\end{aligned}
$$

sign of $\varphi$?

- geostrophy: exact balance between pressure and Coriolis force

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0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\Omega v \sin\varphi \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\Omega u \sin\varphi
\end{aligned}
$$

sign of $\varphi$?

- thermal wind: combine with hydrostatic balance $0 = -\partial p/\partial z - g\rho$

$$
\begin{aligned}
0 &= \frac{g}{\rho_0}\frac{\partial}{\partial x}\rho + 2\Omega\frac{\partial v}{\partial z}\sin\varphi \\
0 &= \frac{g}{\rho_0}\frac{\partial}{\partial y}\rho - 2\Omega\frac{\partial u}{\partial z}\sin\varphi
\end{aligned}
$$

lateral density gradients are related to vertical shear of $u$ and $v$

$\rightarrow$ "dynamical method" to determine ocean currents

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} \quad = \quad -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

- consider third momentum equation

$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + 2u \cos\phi - \frac{\rho g}{\rho_0}$$

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} \quad = \quad -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

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$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + 2u \cos \phi - \frac{\rho g}{\rho_0}$$

- scaling yields

$$\frac{W}{T} \;,\; \left( \frac{UW}{L} , \frac{W^2}{H} \right) \;\sim\; \frac{P}{\rho_0 H} \;,\; \Omega U \;,\; \frac{\varrho g}{\rho_0}$$

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} \quad = \quad -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

- consider third momentum equation

$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + 2u \cos \phi - \frac{\rho g}{\rho_0}$$

- scaling yields

$$\frac{W}{T} \;,\; \left( \frac{UW}{L} \;,\; \frac{W^2}{H} \right) \quad \sim \quad \frac{P}{\rho_0 H} \;,\; \Omega U \;,\; \frac{\varrho g}{\rho_0}$$

- use $W = \delta U$ and $T = L/U$ and scaling for $P = \rho_0 L \Omega U$

$$\frac{\delta U^2}{L} \;,\; \frac{\delta U^2}{L} \quad \sim \quad \frac{L \Omega U}{H} \;,\; \Omega U \;,\; \frac{\varrho g}{\rho_0}$$

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- consider third momentum equation

$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + 2u\cos\phi - \frac{\rho g}{\rho_0}$$

- scaling yields

$$\frac{\delta U^2}{L} \;,\; \frac{\delta U^2}{L} \;\sim\; \frac{L\Omega U}{H} \;,\; \Omega U \;,\; \frac{\varrho g}{\rho_0}$$

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- consider third momentum equation

$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + 2u \cos\phi - \frac{\rho g}{\rho_0}$$

- scaling yields

$$\frac{\delta U^2}{L} \;,\; \frac{\delta U^2}{L} \;\sim\; \frac{L\Omega U}{H} \;,\; \Omega U \;,\; \frac{\varrho g}{\rho_0}$$

- now multiply with $\delta$ and divide by $\Omega U$

$$\frac{\delta^2 U^2}{L\Omega U} = \delta^2 Ro \;,\; \frac{\delta^2 U^2}{L\Omega U} = \delta^2 Ro \;\sim\; 1 \;,\; \delta \;,\; \frac{\delta \varrho g}{\rho_0 \Omega U} \sim 1$$

all magnitudes are now relative to vertical pressure force

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- consider third momentum equation

$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + 2u\cos\phi - \frac{\rho g}{\rho_0}$$

- scaling yields

$$\frac{\delta U^2}{L} \;,\; \frac{\delta U^2}{L} \;\sim\; \frac{L\Omega U}{H} \;,\; \Omega U \;,\; \frac{\varrho g}{\rho_0}$$

- now multiply with $\delta$ and divide by $\Omega U$

$$\frac{\delta^2 U^2}{L\Omega U} = \delta^2 Ro \;,\; \frac{\delta^2 U^2}{L\Omega U} = \delta^2 Ro \;\sim\; 1 \;,\; \delta \;,\; \frac{\delta \varrho g}{\rho_0 \Omega U} \sim 1$$

  all magnitudes are now relative to vertical pressure force
- all terms except $\partial p/\partial z$ and gravity are $O(\delta)$ or smaller

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- consider third momentum equation

$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + 2u\cos\phi - \frac{\rho g}{\rho_0}$$

- scaling yields

$$\frac{\delta U^2}{L} \;,\; \frac{\delta U^2}{L} \;\sim\; \frac{L\Omega U}{H} \;,\; \Omega U \;,\; \frac{\varrho g}{\rho_0}$$

- now multiply with $\delta$ and divide by $\Omega U$

$$\frac{\delta^2 U^2}{L\Omega U} = \delta^2 Ro \;,\; \frac{\delta^2 U^2}{L\Omega U} = \delta^2 Ro \;\sim\; 1 \;,\; \delta \;,\; \frac{\delta \varrho g}{\rho_0 \Omega U} \sim 1$$

  all magnitudes are now relative to vertical pressure force
- all terms except $\partial p/\partial z$ and gravity are $O(\delta)$ or smaller
- since $\delta \ll 1$ neglect all terms except $\partial p/\partial z$ and gravity
  $\rightarrow$ hydrostatic approximation

## Summary hydrostatic approximation

- momentum equation in Boussinesq

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\nabla} \Phi$$

with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- becomes

$$\begin{aligned}
\rho_0 \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + 2\rho_0 \Omega v \sin\phi \\
\rho_0 \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} - 2\rho_0 \Omega u \sin\phi \\
0 &= -\frac{\partial p}{\partial z} - g\rho
\end{aligned}$$

acceleration, advection, etc in 3. momentum equation neglected

- $f = 2\Omega \sin\phi$ is the Coriolis parameter
- other equations are unchanged $\rightarrow$ primitive equations

## Planetary geostrophic approximation

- momentum equation in Boussinesq

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\nabla} \Phi$$

  with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- becomes for Rossby number $Ro \ll 1$

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + 2\rho_0 \Omega v \sin\phi \\ 0 &= -\frac{\partial p}{\partial y} - 2\rho_0 \Omega u \sin\phi \\ 0 &= -\frac{\partial p}{\partial z} - g\rho \end{aligned}$$

  acceleration, advection, etc in all momentum equations neglected

- $f = 2\Omega \sin\phi$ is the Coriolis parameter
- other equations are unchanged