Lecture # 2

Approximations and simplifications

- Boussinesq approximation
- Hydrostatic approximation

- Continuity equation or conservation of mass

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})$$

- introduce scaled variables with primes

$$\rho = \rho_0 \rho \quad , \quad \boldsymbol{u} = U\boldsymbol{u}' \quad , \quad \boldsymbol{x} = L\boldsymbol{x}' \quad , \quad t = Tt'$$

  with the dimensionless functions ρ', $\boldsymbol{u}'$, etc of $O(1)$
  with constants ρ_0, U taking dimensions and magnitudes
  and with $\partial/\partial t = (1/T)\partial/\partial t'$, and $\partial/\partial x = (1/L)\partial/\partial x'$, etc

- this yields

$$\begin{aligned} \frac{\rho_0}{T}\frac{\partial \rho'}{\partial t'} &= -\frac{\rho_0 U}{L}\boldsymbol{\nabla}' \cdot (\rho' \boldsymbol{u}') \\ \frac{\partial \rho'}{\partial t'} &= -\frac{UT}{L}\boldsymbol{\nabla}' \cdot (\rho' \boldsymbol{u}') \end{aligned}$$

- now forget all primes

$$\frac{\partial \rho}{\partial t} = -\frac{UT}{L}\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})$$

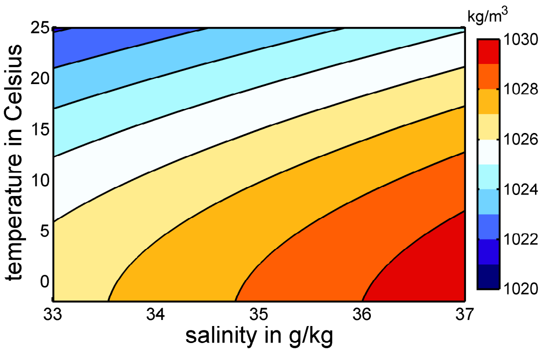
- equation of state for seawater

$$\rho = \rho(S, T, p)$$

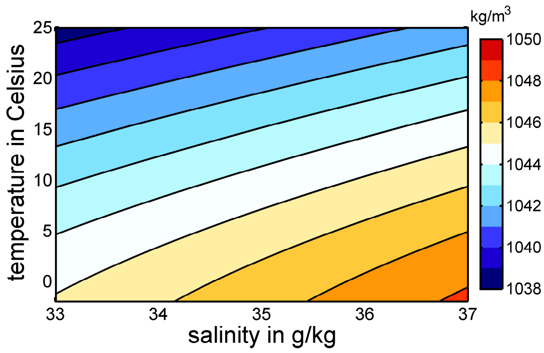
function of salinity S, temperature T and pressure p

- no analytical (exact) expressions for the function $\rho(S, T, p)$
- empirical expressions with relative accuracy of $(3-5) \times 10^{-6}$
$\rightarrow$ TEOS: http://www.teos-10.org/

$\rho(S, T, p = 0\,\mathrm{Pa})$



$\rho(S, T, p = 1000\,\mathrm{dbar} = 10^7\,\mathrm{Pa})$



- Continuity equation or conservation of mass

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})$$

- better scaling than $\rho = \rho_0 \rho'$ is given by $\rho = \rho_0 + \varrho \rho'$
with a large mean value $\rho_0 = 1000\,\mathrm{kg/m^3}$
plus small variations with magnitude $\varrho = 10\,\mathrm{kg/m^3}$
- this yields in the continuity equation

$$\begin{aligned} \frac{1}{T}\frac{\partial}{\partial t'}(\rho_0 + \varrho\rho') &= -\frac{U}{L}\boldsymbol{\nabla}' \cdot ((\rho_0 + \varrho\rho')\boldsymbol{u}') \\ \frac{\partial \rho'}{\partial t'} &= -\frac{\rho_0}{\varrho}\frac{UT}{L}\boldsymbol{\nabla}' \cdot \boldsymbol{u}' - \frac{UT}{L}\boldsymbol{\nabla}' \cdot (\rho' \boldsymbol{u}') \end{aligned}$$

- since $\rho_0/\varrho \gg 1$ and rest of $O(1)$ (as long as $UT/L \geq O(1)$)
it follows that

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \;\rightarrow\; \boldsymbol{\nabla} \cdot \boldsymbol{u} \approx 0 \;:\quad \text{Boussinesq approximation}$$

mass conservation is replaced by volume conservation

- momentum equation (without friction and tides)

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho\boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\nabla} \Phi$$

- using $\rho = \rho_0 + \varrho\rho'$ yields

$$(\rho_0 + \varrho\rho')\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2(\rho_0 + \varrho\rho')\boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho')\boldsymbol{\nabla}\phi$$

- consider 3. component for $\boldsymbol{u} = 0$ and $\phi = gz$

$$\frac{\partial p}{\partial z} = -(\rho_0 + \varrho\rho')g \approx -\rho_0 g$$

- motivates to set $p \equiv p_0(z) + p'$ with $p_0 \gg p'$ and to set

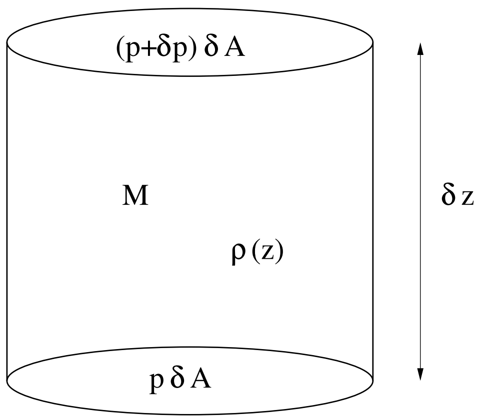
$$\frac{\partial p_0}{\partial z} \equiv -\rho_0 g$$

  hydrostatic balance of p_0 with constant density ρ_0

- momentum equation becomes

$$(\rho_0 + \varrho\rho')\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}(p_0(z) + p') - 2(\rho_0 + \varrho\rho')\boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho')\boldsymbol{\nabla}\phi$$

- consider a column of the ocean



- pressure force at bottom

$$F_B = p\delta A$$

- pressure force at top

$$F_T = -(p + \delta p)\delta A \approx -(p + \frac{\partial p}{\partial z}\delta z)\delta A$$

  (positive upward)

- mass of cylinder is $M = \rho\delta A\delta z$ and gravity force is $F_g = -gM$
- if no other forces act and column does not accelerate

$$F_B + F_T + F_g = 0 \;\rightarrow\; p\delta A - (p + \frac{\partial p}{\partial z}\delta z)\delta A - g\rho\delta A\delta z = 0$$

  or

$$-\frac{\partial p}{\partial z} - g\rho = 0 \;\rightarrow\; \text{hydrostatic balance}$$

- momentum equation becomes

$$(\rho_0 + \varrho\rho')\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}(p_0(z) + p') - 2(\rho_0 + \varrho\rho')\boldsymbol{\Omega} \times \boldsymbol{u} - (\rho_0 + \varrho\rho')\boldsymbol{\nabla}\phi$$

- since by construction

$$-\frac{\partial p_0}{\partial z} - g\rho_0 = 0 \quad \text{or} \quad -\boldsymbol{\nabla} p_0(z) - \rho_0\boldsymbol{\nabla}\phi = 0$$

background gravity and vertical pressure gradient completely drop

- since $\rho_0 \gg \varrho$ momentum equation further simplifies to

$$\rho_0\frac{D\boldsymbol{u}}{Dt} \approx -\boldsymbol{\nabla} p' - 2\rho_0\boldsymbol{\Omega} \times \boldsymbol{u} - \varrho\rho'\boldsymbol{\nabla}\phi$$

finally set $p' \to p$ and $\varrho\rho' \to \rho$

but remember that pressure p and density ρ are now *perturbations*

- salinity equation with $\rho = \rho_0 + \varrho\rho'$ becomes

$$(\rho_0 + \varrho\rho')\frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S \quad \to \quad \rho_0\frac{DS}{Dt} \approx -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S$$

and similar for temperature

Conservation laws in Boussinesq approximation

- momentum equation

$$\rho_0\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} - \rho\boldsymbol{\nabla}(\Phi + \Phi_{tide})$$

with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- continuity equation

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

- salt conservation equation

$$\rho_0\frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S$$

- conservative temperature equation

$$\rho_0\frac{D\Theta}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term}$$

- equation of state with conservative temperature as state variable

$$\rho = \rho(S, \Theta, p_0)$$

- scale continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \rightarrow \quad \frac{U}{L}\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}\right)+\frac{W}{H}\frac{\partial w'}{\partial z'}=0$$

  with lateral scale L and vertical scale H
  and lateral velocity scale U and vertical velocity scale W
- for $W/H \gg U/L$ it follows that $\partial w/\partial z = 0$
  such that $w = 0$ considering bottom or top boundaries
  $\rightarrow$ scaling becomes inconsistent
- only cases $W/H \sim U/L$ or $W/H \ll U/L$ are possible
- now define aspect ratio $\delta = H/L$ ("deepness" of the flow) with

$$W = UH/L = \delta U$$

  which means that for $\delta \sim 1 \ \rightarrow \ W \sim U$ and for $\delta \ll 1 \ \rightarrow \ W \ll U$
- since $\delta \ll 1$ for large-scale flow in the ocean $W \ll U$
  i.e. shallow water yields small (but still important!) w

- aspect ratio $\delta = H/L \ll 1$, $W = \delta U$
- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} \quad = \quad -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

- consider first momentum equation

$$\frac{\partial u}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla} u = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - 2\Omega(w\cos\varphi - v\sin\varphi)$$

- scaling of each term yields

$$\frac{U}{T} \ , \ \left(\frac{U^2}{L} , \frac{WU}{H}\right) \ \sim \ \frac{P}{\rho_0 L} \ , \ \Omega W \ , \ \Omega U$$

  divide by ΩU to get magnitudes relative to (vertical) Coriolis force

$$\frac{1}{T\Omega} \ , \ \frac{U}{L\Omega} \ \sim \ \frac{P}{\rho_0 L \Omega U} \ , \ \delta \ll 1 \ , \ 1$$

- aspect ratio $\delta = H/L \ll 1$, $W = \delta U$
- consider first momentum equation

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - 2\Omega(w \cos\varphi - v \sin\varphi)$$

- scaling yields

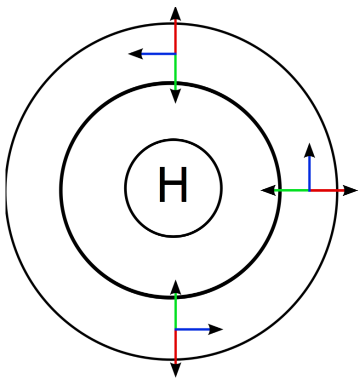
$$\frac{1}{T\Omega} \ , \ \frac{U}{L\Omega} \ \sim \ \frac{P}{\rho_0 L\Omega U} \ , \ \delta \ll 1 \ , \ 1$$

- set $T = L/U$ and define Rossby number $Ro = U/(L\Omega)$

$$Ro \ , \ Ro \ \sim \ \frac{P}{\rho_0 L\Omega U} \ , \ \delta \ll 1 \ , \ 1$$

- Ro compares momentum advection with Coriolis force for large-scale flow in the ocean $Ro \leq 1$
- assume dominant geostrophic balance: $P/(\rho_0 L\Omega U) = 1$ but still keep terms of $O(Ro)$ (but not those of $O(\delta)$)!

- geostrophy: exact balance between pressure and Coriolis force



$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\Omega v \sin\varphi$$
$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\Omega u \sin\varphi$$

sign of φ?

- thermal wind: combine with hydrostatic balance $0 = -\partial p/\partial z - g\rho$

$$0 = \frac{g}{\rho_0}\frac{\partial}{\partial x}\rho + 2\Omega\frac{\partial v}{\partial z}\sin\varphi$$
$$0 = \frac{g}{\rho_0}\frac{\partial}{\partial y}\rho - 2\Omega\frac{\partial u}{\partial z}\sin\varphi$$

lateral density gradients are related to vertical shear of u and v
$\rightarrow$ "dynamical method" to determine ocean currents

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- momentum equation in rotating frame (no friction, no tides)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \rho \boldsymbol{\nabla} \phi$$

- consider third momentum equation

$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + 2u\cos\phi - \frac{\rho g}{\rho_0}$$

- scaling yields

$$\frac{W}{T} \ , \ \left(\frac{UW}{L} \ , \ \frac{W^2}{H}\right) \ \sim \ \frac{P}{\rho_0 H} \ , \ \Omega U \ , \ \frac{\varrho g}{\rho_0}$$

- use $W = \delta U$ and $T = L/U$ and scaling for $P = \rho_0 L \Omega U$

$$\frac{\delta U^2}{L} \ , \ \frac{\delta U^2}{L} \ \sim \ \frac{L\Omega U}{H} \ , \ \Omega U \ , \ \frac{\varrho g}{\rho_0}$$

- aspect ration $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- consider third momentum equation

$$\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + 2u\cos\phi - \frac{\rho g}{\rho_0}$$

- scaling yields

$$\frac{\delta U^2}{L} \ , \ \frac{\delta U^2}{L} \ \sim \ \frac{L\Omega U}{H} \ , \ \Omega U \ , \ \frac{\varrho g}{\rho_0}$$

- now multiply with δ and divide by ΩU

$$\frac{\delta^2 U^2}{L\Omega U} = \delta^2 Ro \ , \ \frac{\delta^2 U^2}{L\Omega U} = \delta^2 Ro \ \sim \ 1 \ , \ \delta \ , \ \frac{\delta \varrho g}{\rho_0 \Omega U} \sim 1$$

  all magnitudes are now relative to vertical pressure force
- all terms except $\partial p/\partial z$ and gravity are $O(\delta)$ or smaller
- since $\delta \ll 1$ neglect all terms except $\partial p/\partial z$ and gravity
  $\rightarrow$ hydrostatic approximation

Summary hydrostatic approximation

- momentum equation in Boussinesq

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\nabla} \Phi$$

  with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- becomes

$$\begin{aligned} \rho_0 \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + 2\rho_0 \Omega v \sin\phi \\ \rho_0 \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} - 2\rho_0 \Omega u \sin\phi \\ 0 &= -\frac{\partial p}{\partial z} - g\rho \end{aligned}$$

  acceleration, advection, etc in 3. momentum equation neglected

- $f = 2\Omega \sin\phi$ is the Coriolis parameter
- other equations are unchanged $\rightarrow$ primitive equations

Planetary geostrophic approximation

- momentum equation in Boussinesq

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\nabla} \Phi$$

  with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- becomes for Rossby number $Ro \ll 1$

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + 2\rho_0 \Omega v \sin\phi \\ 0 &= -\frac{\partial p}{\partial y} - 2\rho_0 \Omega u \sin\phi \\ 0 &= -\frac{\partial p}{\partial z} - g\rho \end{aligned}$$

  acceleration, advection, etc in all momentum equations neglected

- $f = 2\Omega \sin\phi$ is the Coriolis parameter
- other equations are unchanged