Dynamische und regionale Ozeanographie WS 2014/15

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Lecture # 3

Content dynamical oceanography

1. Introduction
2. Hydrodynamics

   Kinematics, continuity, momentum and thermodynamic equation

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   Gravity waves w/o rotation, Kelvin waves, geostrophic adjustment, Rossby waves, vertical modes, equatorial waves

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   Ekman-layers, -spiral, -transport, -pumping, Sverdrup transport, western boundary currents

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6. Thermohaline circulation

   basic ingredients and dynamics, Stommel-Arons model

Inhalt Regionale Ozeanographie

- Fundamentale Eigenschaften des Meerwassers (The international thermodynamic equation of seawater – 2010, TEOS-10) Calculation and use of thermodynamic properties)
- Methoden der Wassermassenanalyse (Temperatur-, Salzgehalts- und Dichte-Gleichung, Zustandsgleichung, Stirring & Mixing)
- Flüsse an der Grenzfläche Atmosphäre-Ozean (Energiebilanzen Ozean & Atmosphäre)
- Eigenschaften und Dynamik der ozeanischen Deckschicht, Wassermassenformation (Konvektion, Ekman, Langmuir, Trägheitswellen, Scherungsinstabilitäten)
- Subduktion und Dynamik der Warmwassersphäre (Auftrieb, Geostrophie, Vorticityerhaltung, Sverdrup, Subduktion, Ventilierte Thermokline)
- Westliche Randströme (Stommel, Munk, Ro > 0, Rossby-Wellen, Instabilitäten, Randströme im Äquatorbereich)
- Randmeere des Atlantischen Ozeans
- Reg. Oz. des Indischen Ozeans
- Reg. Oz. des Pazifischen Ozeans
- Reg. Oz. des Südlichen Ozeans

(Hier regionale Besonderheiten z.B. Monsunzirkulation, Äquatoriale Zirkulation, El Nino, Form Drag AACC, Schelf-Slope Konvektion, Overflows)

Literature

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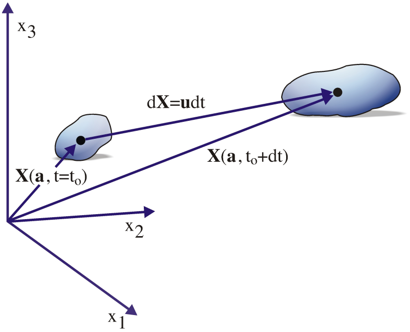
- (Olbers, Willebrand, Eden: Ocean dynamics)

- any fluid is made of small 'water parcels'



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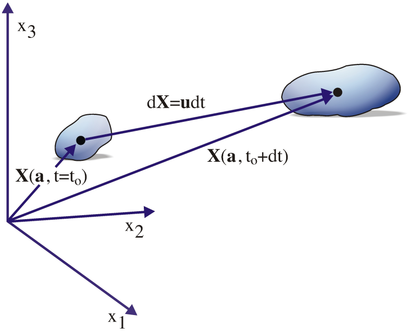

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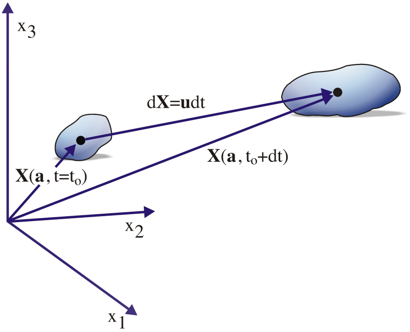

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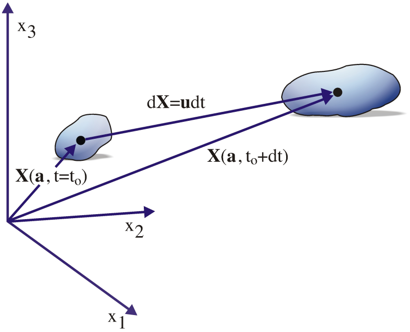

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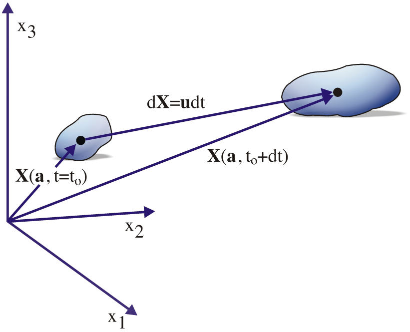

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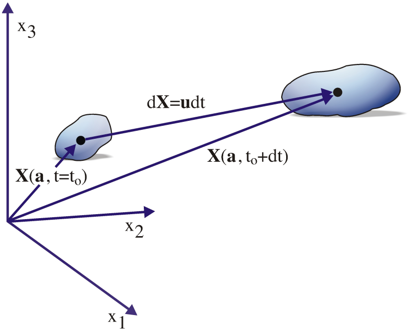
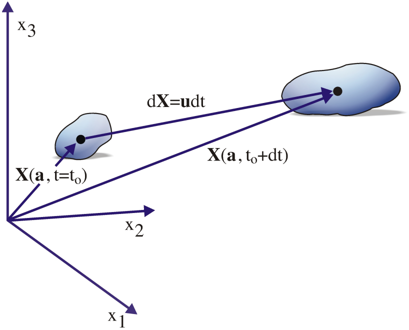
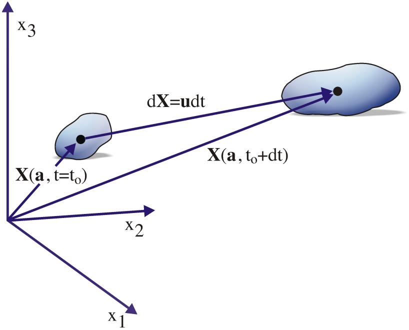
- velocity $\boldsymbol{u}$ is the parcel velocity, a parcel has properties, e.g. C

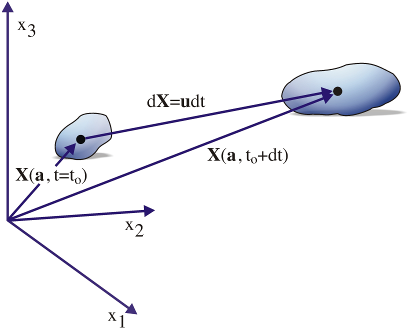



x3
dX=udt
X(a, t=to)
X(a, to+dt)
x2
x1



- consider property $C(t,x,y,z)$ of a water parcel

$$\delta C = \frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial x}\delta x + \frac{\partial C}{\partial y}\delta y + \frac{\partial C}{\partial z}\delta z$$



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  i.e. calculate δC following path of parcel

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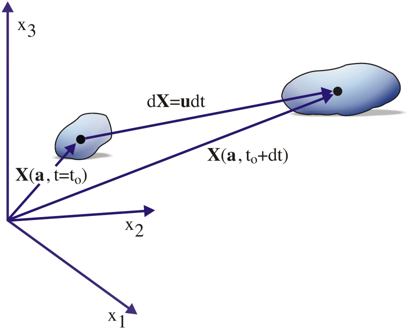

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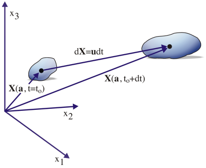
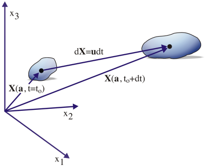
$$\delta C = \frac{\partial C}{\partial t}\delta t + \left(\frac{\partial C}{\partial x}u + \frac{\partial C}{\partial y}v + \frac{\partial C}{\partial z}w\right)\delta t \;\rightarrow\; \frac{\delta C}{\delta t} = \frac{\partial C}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} C$$



- Euler's relation

$$\frac{\delta C}{\delta t} \rightarrow \left(\frac{\partial}{\partial t} C\right)_{parcel} = \frac{\partial}{\partial t} C + \boldsymbol{u} \cdot \boldsymbol{\nabla} C \equiv \frac{D}{Dt} C$$

- $D/Dt = \partial/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$ is often called 'material' or 'substantial' derivative



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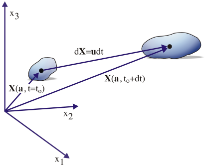

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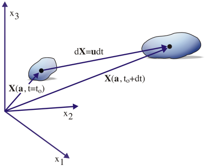

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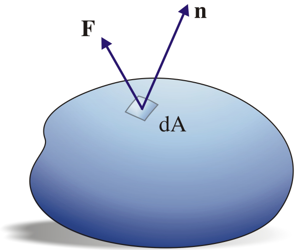
- Lagrangian frameworks uses left hand side of DC/Dt

  Eulerian framework uses right hand side of DC/Dt

  both are equivalent but Eulerian framework is often more convenient

- consider volume V, *fixed* in space and bounded by a surface A
- and a scalar fluid property C concentration
  (in units of C per kg sea water or ρC in units of C per m^3)



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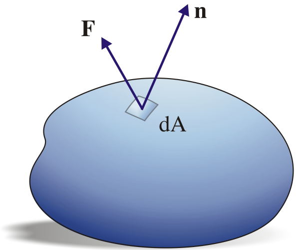
- total amount of the property C in V is given by

$$\int_V \rho C \, dV$$

and may change in time by two ways:



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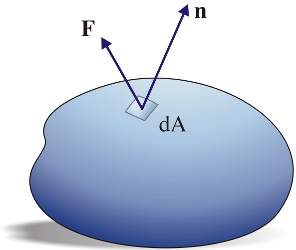
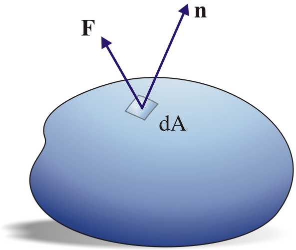
  and may change in time by two ways:
  - by a flux across surface A
  - by an interior source or sink



- consider volume V, *fixed* in space and bounded by a surface A
- outward transport across A

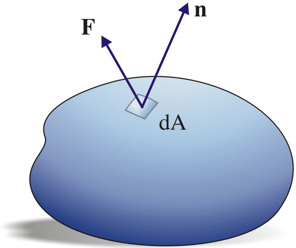
$$\oint_A (\rho C \boldsymbol{u} + \boldsymbol{J}) \cdot d\boldsymbol{A}$$

- by transport by water parcels, an "advective" part $\rho C \boldsymbol{u}$
- and by a non-advective flux $\boldsymbol{J}$ which is everything else, e.g diffusion, heat conduction, radiation etc.



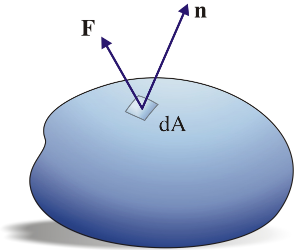
- consider volume V, *fixed* in space and bounded by a surface A
- interior sources/sinks Q (units of C per time and volume), e.g. heat sources, radioactive decay, chemical reaction,

$$\int_V Q\, dV$$



- consider volume V, *fixed* in space and bounded by a surface A
- total rate of change of the C-content in the volume

$$\frac{\partial}{\partial t}\int_V \rho C\, dV = -\oint_A (\rho C \boldsymbol{u} + \boldsymbol{J}) \cdot d\boldsymbol{A} + \int_V Q\, dV$$



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- the surface integral may be rewritten with Gauss law as

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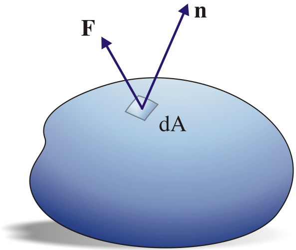

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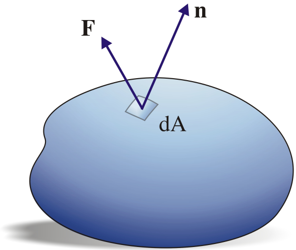
- for fixed volume $\partial/\partial t$ and integral commute



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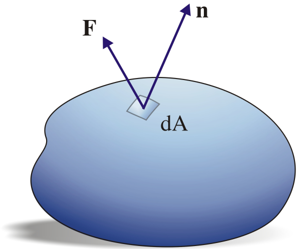
$$\int_V \left[\frac{\partial}{\partial t}\rho C + \boldsymbol{\nabla}\cdot(\rho C\boldsymbol{u} + \boldsymbol{J}) - Q\right] dV = 0$$

- since this holds for *arbitrary* volume, the integrand has to vanish

$$\frac{\partial}{\partial t}\rho C + \boldsymbol{\nabla}\cdot(\rho C\boldsymbol{u} + \boldsymbol{J}) - Q = 0$$

which is the general conservation equation in flux form



- general conservation law in flux form

$$\frac{\partial}{\partial t}\rho C = -\boldsymbol{\nabla} \cdot (\rho C \boldsymbol{u} + \boldsymbol{J}) + Q$$

- take $C = 1($ kg /kg sea water$)$, $\rightarrow \rho C$ becomes total mass per m^3

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- mass conservation or continuity equation

$$\frac{\partial}{\partial t}\rho \quad = \quad -\boldsymbol{\nabla} \cdot \rho \boldsymbol{u}$$

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- possible to rewrite flux form (above) to parcel form

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$$\frac{D\rho}{Dt} = \frac{D}{Dt}v^{-1} = -\frac{1}{v^2}\frac{Dv}{Dt} = -\frac{1}{v}\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

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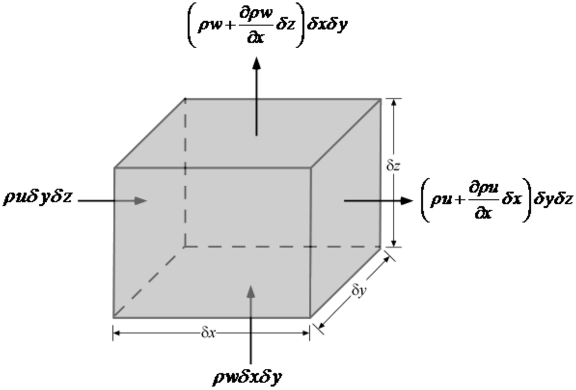
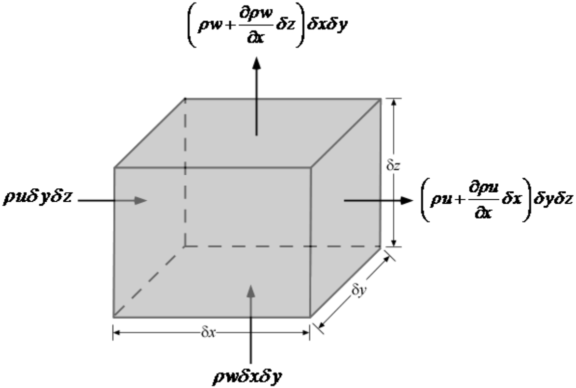
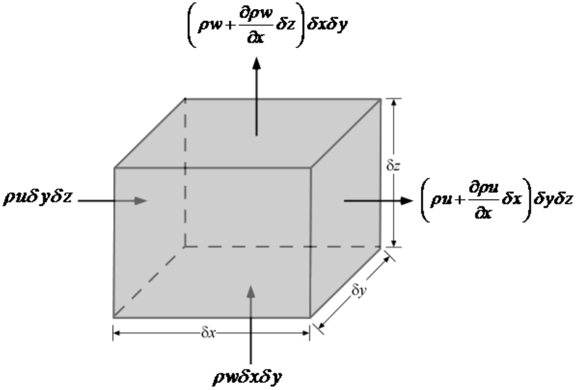
parcel form of continuity equation


$\left(\rho w+\frac{\partial\rho w}{\partial x}\delta z\right)\delta x\delta y$
$\rho u\delta y\delta z$
δz
$\left(\rho u+\frac{\partial\rho u}{\partial x}\delta x\right)\delta y\delta z$
δy
δx
$\rho w\delta x\delta y$



- divergence of mass flux F in x direction

$$F^x(x_0 + \delta x) - F^x(x_0) = \left(\rho u + \frac{\partial \rho u}{\partial x}\delta x + \cancel{O(\delta x^2)}\right)\delta y \delta z - \rho u \delta y \delta z$$



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- rate of change of mass

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- divergence of mass flux F in y direction

$$F^y(y_0 + \delta y) - F^y(y_0) = \frac{\partial \rho v}{\partial y}\delta x \delta y \delta z$$

and similar for z

- divergence of mass flux F in x direction

$$F^x(x_0 + \delta x) - F^x(x_0) = (\rho u + \frac{\partial \rho u}{\partial x}\delta x)\delta y \delta z - \rho u \delta y \delta z$$

- rate of change of mass

$$\frac{\delta M}{\delta t} = \frac{\partial}{\partial t}(\rho \delta x \delta y \delta z) = \frac{\partial \rho}{\partial t}\delta x \delta y \delta z$$

- divergence of mass flux F in y direction

$$F^y(y_0 + \delta y) - F^y(y_0) = \frac{\partial \rho v}{\partial y}\delta x \delta y \delta z$$

and similar for z

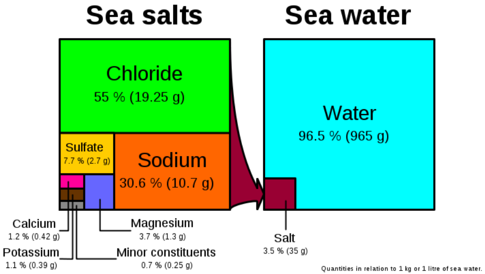
- mass change is balanced by flux divergences

$$\frac{\delta M}{\delta t} = \frac{\partial \rho}{\partial t}\delta x \delta y \delta z = -\left(\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w\right)\delta x \delta y \delta z$$

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \rho \boldsymbol{u}$$

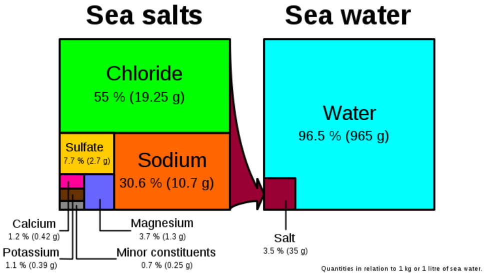
which is again the flux form of the continuity equation

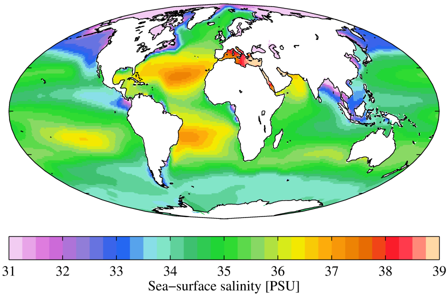
- seawater is mixture of pure water and various salts



- seawater is mixture of pure water and various salts
- combine all salts into one variable :

  s is salt concentration in kg salt/kg sea water
- $s = 0.035$ is a typical value for the open ocean





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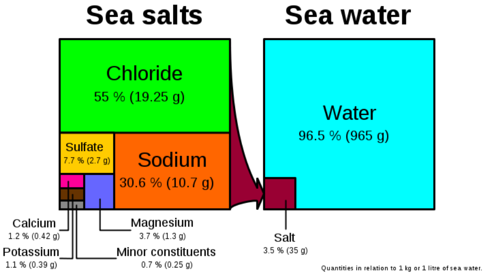
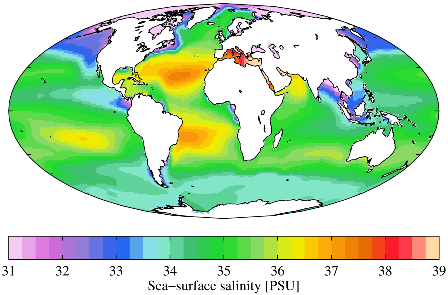
- salinity is $S = 1000s$ such that s=0.035 kg salt/kg sea water corresponds to $S = 35\mathrm{g}/\mathrm{kg}$





- general conservation law in flux form

$$\frac{\partial}{\partial t}\rho C = -\boldsymbol{\nabla}\cdot(\rho C\boldsymbol{u} + \boldsymbol{J}) + Q$$

- salt conservation equation with $C = S$, $Q = 0$ but $\boldsymbol{J} = \boldsymbol{J}_S$

$$\frac{\partial}{\partial t}\rho S = -\boldsymbol{\nabla}\cdot(\rho S\boldsymbol{u} + \boldsymbol{J}_S)$$

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- rewrite to parcel form given by

$$\rho\frac{\partial S}{\partial t} + S\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot\rho S\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{J}_S$$

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\rho\frac{DS}{Dt} &= -\boldsymbol{\nabla}\cdot\boldsymbol{J}_S
\end{aligned}$$

using continuity equation $\partial\rho/\partial t = -\boldsymbol{\nabla}\cdot\rho\boldsymbol{u}$ times S

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\rho\frac{DS}{Dt} &= -\boldsymbol{\nabla}\cdot\boldsymbol{J}_S = \boldsymbol{\nabla}\cdot\kappa_S\boldsymbol{\nabla}S
\end{aligned}$$

  using continuity equation $\partial\rho/\partial t = -\boldsymbol{\nabla}\cdot\rho\boldsymbol{u}$ times S

- specify $\boldsymbol{J}_S$ as 'downgradient' diffusive flux $\rightarrow$ $\boldsymbol{J}_S = -\kappa_S\boldsymbol{\nabla}S$
  with molecular diffusivity for salinity $\kappa_S \approx 1.2\times10^{-9}\,\mathrm{m}^2/\mathrm{s}$

- general conservation equation

$$\frac{\partial}{\partial t}\rho C = -\boldsymbol{\nabla}\cdot(\rho C\boldsymbol{u}+\boldsymbol{J})+Q \ , \ \rho\frac{DC}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{J}+Q$$

flux form and parcel form

- general conservation equation

$$\frac{\partial}{\partial t}\rho C = -\boldsymbol{\nabla}\cdot(\rho C\boldsymbol{u}+\boldsymbol{J})+Q \ , \ \rho\frac{DC}{Dt}=-\boldsymbol{\nabla}\cdot\boldsymbol{J}+Q$$

flux form and parcel form

- flux form for momentum component $u_i = C$

$$\frac{\partial}{\partial t}\rho u_i = -\boldsymbol{\nabla}\cdot\left(\rho u_i\boldsymbol{u}+\boldsymbol{J}^{(i)}\right)+Q_i$$

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$$\rho\frac{Du_i}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{J}^{(i)}+Q_i = -\frac{\partial}{\partial x_j}J_j^{(i)}+Q_i$$

Newton's law for parcels → right hand side are forces (per volume)

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  Newton's law for parcels → right hand side are forces (per volume)

- in vector form and with (stress) tensor $-\Pi_{ji} = J_j^{(i)}$

$$\rho\frac{D\boldsymbol{u}}{Dt}=\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}+\boldsymbol{Q}$$

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Newton's law for parcels $\rightarrow$ right hand side are forces (per volume)

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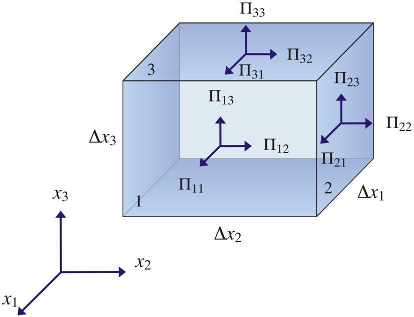
$$\rho\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\nabla}\cdot\boldsymbol{\Pi}+\boldsymbol{Q}$$

- $\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}$ and $\boldsymbol{Q}$ are forces (per volume) acting on the water parcel

- the stress tensor $\mathbf{\Pi}$ or Π_{ji} with

$$dF_i = dA\, n_j\, \Pi_{ji} \quad \text{or} \quad d\boldsymbol{F} = d\boldsymbol{A} \cdot \mathbf{\Pi} \;,\; d\boldsymbol{A} = \boldsymbol{n} dA$$

- $n_j \Pi_{ji}$ is the i–component of the force per unit area on the area perpendicular to $\boldsymbol{n}$



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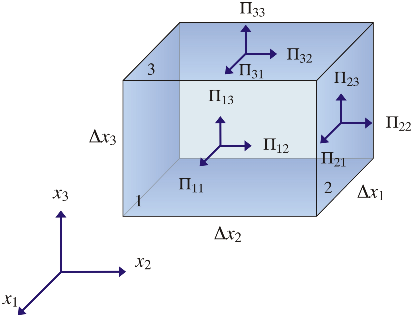
- Π_{ji} stands for the i–component of the force per unit area (stress) on the area perpendicular to the j–axis



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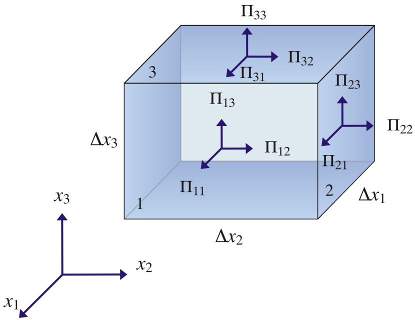
- Π_{11}, Π_{22} and Π_{33} are the normal stresses, rest are tangential stresses



- conservation of momentum (Navier-Stokes equations)

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \frac{\partial}{\partial t}\boldsymbol{u} + \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \quad = \quad \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} + \boldsymbol{Q}$$

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$$\rho \frac{\partial}{\partial t} u_i + \rho u_j \frac{\partial}{\partial x_j} u_i = \frac{\partial}{\partial x_j} \Pi_{ji} + Q_i$$

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$$
\begin{aligned}
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$$

- diagonal elements of $\boldsymbol{\Pi}$ acting normal to the corresponding surface are *normal stresses*

  off-diagonal elements of $\boldsymbol{\Pi}$ acting tangential are the *tangential stresses*

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- the mean normal inward stress is the (mechanical) *pressure*

$$p = -\frac{1}{3} \left(\Pi_{11} + \Pi_{22} + \Pi_{33} \right) = -\frac{1}{3} \Pi_{ii} = -\frac{1}{3} \mathrm{tr}\, \boldsymbol{\Pi}$$

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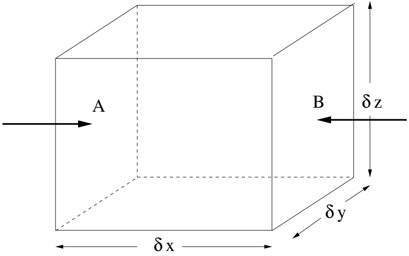
- decompose $\boldsymbol{\Pi}$ into isotropic pressure part and remainder

$$\Pi_{ij} = -p\delta_{ij} + \Sigma_{ij} \qquad \text{or} \qquad \boldsymbol{\Pi} = -p\boldsymbol{I} + \boldsymbol{\Sigma} \ \text{ with } \ \boldsymbol{\nabla} \cdot p\boldsymbol{I} = \boldsymbol{\nabla} p$$

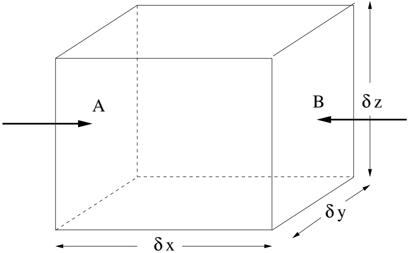
  with the frictional tensor $\boldsymbol{\Sigma}$ with vanishing trace Σ_{ii}



- pressure force on side A and side B

$$F_A = p(x_0)\delta y \delta z \quad , \quad F_B = -p(x_0 + \delta x)\delta y \delta z$$

and similar for all other sides



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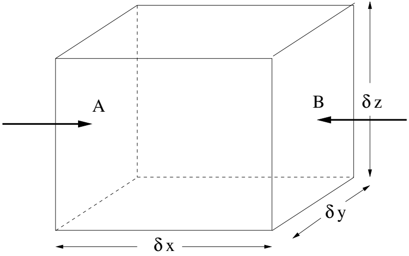
$$F^{(i)} = -\frac{\partial p}{\partial x_i}\delta x \delta y \delta z$$



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  the net force in x_i-direction is given by

$$F^{(i)} = -\frac{\partial p}{\partial x_i}\delta x \delta y \delta z \;\rightarrow\; \frac{F^{(i)}}{\delta x \delta y \delta z} = f^{(i)} = \frac{\partial p}{\partial x_i} \;\rightarrow\; \boldsymbol{f} = -\boldsymbol{\nabla} p$$

- balance of momentum in parcel form

$$\rho \frac{D\boldsymbol{u}}{Dt} = \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} + \boldsymbol{f}^{v} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} + \boldsymbol{Q}$$

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- Newtonian fluid: relation between friction and velocity shear

$$\Sigma_{ij} = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_\ell}{\partial x_\ell} \delta_{ij} \right)$$

with the (dynamical) viscosity ν

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- Navier-Stokes equation for Newtonian fluid (and constant ν)

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- remaining forces in $\boldsymbol{Q}$ are gravity, centrifugal, and Coriolis force

$$\boldsymbol{Q} = -2\rho \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\nabla}(\Phi + \Phi_{tide})$$

maybe also surface tension, electromagnetic forces, etc

- conservation equation for in situ temperature T

$$\rho c_p \frac{DT}{Dt} = \alpha T \frac{Dp}{Dt} + \frac{\partial H}{\partial S} \boldsymbol{\nabla} \cdot \boldsymbol{J}_S - \boldsymbol{\nabla} \cdot \boldsymbol{J}_H + \rho\epsilon$$

with enthalpy H, specific heat $c_p = \partial H / \partial T$, thermal expansion coefficient $\alpha = -1/\rho\, \partial\rho/\partial T$, kinetic energy dissipation $\rho\epsilon = \Sigma_{ij}^2$ and molecular diffusive enthalpy flux $\boldsymbol{J}_H$

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- assume adiabatic conditions, i.e. $\boldsymbol{J}_S = 0$, $\boldsymbol{J}_H = 0$ and $\epsilon = 0$

$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{Dp}{Dt} = 0$$

- conservation equation for in situ temperature T

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with adiabatic lapse rate $\Gamma = \alpha T / (\rho c_p)$

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- conservation equation for in situ temperature T

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- in situ temperature is not "conserved"
- changes in temperature and pressure are related by $dT = \Gamma dp$

- conservation equation for in situ temperature T

$$\rho c_p \frac{DT}{Dt} = \alpha T \frac{Dp}{Dt} + \frac{\partial H}{\partial S} \boldsymbol{\nabla} \cdot \boldsymbol{J}_S - \boldsymbol{\nabla} \cdot \boldsymbol{J}_H + \rho\epsilon$$

with enthalpy H, specific heat $c_p = \partial H/\partial T$, thermal expansion coefficient $\alpha = -1/\rho\, \partial\rho/\partial T$, kinetic energy dissipation $\rho\epsilon = \Sigma_{ij}^2$ and molecular diffusive enthalpy flux $\boldsymbol{J}_H$

- assume adiabatic conditions, i.e. $\boldsymbol{J}_S = 0$, $\boldsymbol{J}_H = 0$ and $\epsilon = 0$

$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{Dp}{Dt} = 0 \quad \text{or} \quad \frac{DT}{Dt} = \Gamma \frac{Dp}{Dt}$$

with adiabatic lapse rate $\Gamma = \alpha T/(\rho c_p)$

- in situ temperature is not "conserved"
- changes in temperature and pressure are related by $dT = \Gamma dp$
- typical value is $\Gamma \approx 10^{-8}\,\mathrm{K/Pa} = 10^{-4}\,\mathrm{K/dbar} \sim 0.1\,\mathrm{K/km}$

- use equation for "conservative temperature" Θ instead

$$\rho \frac{D\Theta}{Dt} = \left(\frac{\theta}{T} \frac{\partial H}{\partial S} - \frac{\partial H^0}{\partial S} \right) \boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_S}{c_p^\star} + \frac{\theta}{T} \left(-\boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_H}{c_p^\star} + \rho \frac{\epsilon}{c_p^\star} \right)$$

  with "potential enthalpy" $H^0 = H(p = p_{ref})$, reference specific heat $c_p^\star = const$, "potential temperature" $\theta = \partial H^0 / \partial \eta$ (with entropy η)

- now assume $\theta / T \approx 1$ with relative error of 10^{-3}

- use equation for "conservative temperature" Θ instead

$$\rho \frac{D\Theta}{Dt} = \left(\frac{\theta}{T} \frac{\partial H}{\partial S} - \frac{\partial H^0}{\partial S} \right) \boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_S}{c_p^\star} + \frac{\theta}{T} \left(-\boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_H}{c_p^\star} + \rho \frac{\epsilon}{c_p^\star} \right)$$

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- neglect effect of salt fluxes compared to heat flux term $\boldsymbol{J}_H$

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$$\rho \frac{D\Theta}{Dt} = \left(\frac{\theta}{T}\frac{\partial H}{\partial S} - \frac{\partial H^0}{\partial S}\right) \boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_S}{c_p^\star} + \frac{\theta}{T}\left(-\boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_H}{c_p^\star} + \rho \frac{\epsilon}{c_p^\star}\right)$$

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with "potential enthalpy" $H^0 = H(p = p_{ref})$, reference specific heat $c_p^\star = const$, "potential temperature" $\theta = \partial H^0 / \partial \eta$ (with entropy η)

- now assume $\theta / T \approx 1$ with relative error of 10^{-3}
- neglect effect of salt fluxes compared to heat flux term $\boldsymbol{J}_H$
- neglect effect of dissipation compared to heat flux term
- get temperature equation containing the divergence of $\boldsymbol{J}_H$

$$\rho \frac{D\Theta}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term}$$

with $\boldsymbol{J}_\Theta = \boldsymbol{J}_H / c_p^\star \rightarrow$ neglect small source term

- use equation for "conservative temperature" Θ instead

$$\rho \frac{D\Theta}{Dt} = \left(\frac{\theta}{T} \frac{\partial H}{\partial S} - \frac{\partial H^0}{\partial S} \right) \boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_S}{c_p^\star} + \frac{\theta}{T} \left(-\boldsymbol{\nabla} \cdot \frac{\boldsymbol{J}_H}{c_p^\star} + \rho \frac{\epsilon}{c_p^\star} \right)$$

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- get temperature equation containing the divergence of $\boldsymbol{J}_H$

$$\rho \frac{D\Theta}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term} \approx \boldsymbol{\nabla} \cdot \kappa_\Theta \boldsymbol{\nabla} \Theta$$

with $\boldsymbol{J}_\Theta = \boldsymbol{J}_H / c_p^\star \rightarrow$ neglect small source term

- specify $\boldsymbol{J}_\Theta$ as 'downgradient' diffusive flux $\rightarrow \boldsymbol{J}_\Theta = -\kappa_\Theta \boldsymbol{\nabla} \Theta$
  with molecular diffusivity for heat (enthalpy) $\kappa_\Theta \approx 1.4 \times 10^{-7}\, \mathrm{m}^2/\mathrm{s}$

Summary conservation laws

- momentum equation

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} + \boldsymbol{Q} \quad , \quad \boldsymbol{Q} = -2\rho\boldsymbol{\Omega} \times \boldsymbol{u} - \rho\boldsymbol{\nabla}(\Phi + \Phi_{tide})$$

with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- continuity equation

$$\frac{D\rho}{Dt} = -\rho\boldsymbol{\nabla} \cdot \boldsymbol{u} \quad , \quad \rho\frac{Dv}{Dt} = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

- salt conservation equation

$$\rho\frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S = \boldsymbol{\nabla} \cdot \kappa_S \boldsymbol{\nabla} S$$

- conservative temperature equation

$$\rho\frac{D\Theta}{Dt} \quad = \quad -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term} \approx \boldsymbol{\nabla} \cdot \kappa_\Theta \boldsymbol{\nabla}\Theta$$

- equation of state with conservative temperature as state variable

$$\rho \; = \; \rho(S, \Theta, p)$$